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SOME NEW RESULTS ON DECOMPOSABLE OPERATOR

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***ABSTRACT*−*In this paper the Decomposable Operators on Banach Spaces characterize a single-valued extension property with analytic function by using of Rouche theorem and Dunford's theorem.*

Keywords- Single valued extension property, quasi-nilpotent ,spectral maximal space.

1, INTRODUCTION

C.Foias (1963) for the first time introduced decomposable operator on Banach space. Radjabalipour (1978) furnishes the equivalency of decomposable and 2-decomposable operators with a new approach. Albrecht (1979) emphasized on the application of decomposable operators in integral equations and vice-versa. Lange (1981) introduced equivalent conditions for decomposable operators over and above the conditions given by previous authors. Erdelyi and Wang (1985) discussed some new characterization of such operators. Later on many authors extended decomposable operators with different aspects. Still some new results are lying unclaimed in the domain of decomposable operators. In this paper we gave four new characterizations of decomposable operators. We first developed here a few results that are needed for the characterization of decomposability. The theory is clearly linked to that of single valued extension property for any analytic functions.

Definition: Let X be a Banach space and $B(X)$ the Banach algebra of the linear bounded operators of X then an operator $T \in B(X)$ is said to have the single valued extension property if for any analytic function $f : D_f \to X, D_f \subseteq C$ open with $(\lambda I - T) f(\lambda) = 0$ it results $f(\lambda) = 0$

For an operator $T \in B(X)$ having the single valued extension property and for $x \in X$ we can consider the set $\rho_T(x)$ of elements $\lambda_o \in C$ such that ∃ an analytic function $\lambda \to x(\lambda)$ defined in a neighborhood of λ_0 with values in X, which verifies $(\lambda I - T)x(\lambda) \equiv x$ then According to definition $x(\lambda)$ is unique. Evidently $\rho_T(x)$ is open and $\rho(T) \subseteq \rho_T(x)$ Take $\rho_T(x) = C\rho_T(x)$ and $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ where $F \subseteq C$

Clearly $X_T(F) = X_T\left[F \cap \sigma_{(T)}\right]$

Theorem (1): Let $T \in B(X)$ be an operator having the single valued extension property. Then

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(i) $F_1 \subseteq F_2$ implies $X_T(F_1) \subseteq X_T(F_2)$

(ii) $X_T(F)$ is a linear subspace (not necessarily closed) of X

(iii) $\sigma_T(x) = \phi$ if and only if $x = 0$

(iv) $\sigma_T(A_x) \subseteq \sigma_T(x)$ $\forall A \in B(x)$ with $AT = TA$

(v) $\sigma_T(x_{(\lambda)}) = \sigma_T(x)$ for every $x \in X$ & $\lambda \in \sigma_T(x)$

Theorem (2): Let $T_\alpha \in B(X_\alpha)$ (α=1,2) Then $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ has the single valued extension property if and only if both T_1 and T_2 have this property, moreover $\sigma_{T_1 \oplus T_2}(x_1 \oplus x_2) = \sigma_{T_1}(x_1) \cup \sigma_{T_2}(x_2)$

Proof: Suppose that T_1 and T_2 have the single-valued extension property and let $f = f_1 \oplus f_2$ be an analytic $X_1 \oplus X_2$ valued function defined on an open set G where

$f_1 : G \to X_1$ & $f_2 : G \to X_2$

are analytic functions. Now if

$(\lambda I_1 - T_1) f_1(\lambda) \oplus (\lambda I_2 - T_2) f_2(\lambda) = [\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2)] f(x)$ then

$(\lambda I_1 - T_1) f_1(\lambda) \equiv 0$ & $(\lambda I_2 - T_2) f_2(\lambda) \equiv 0$

Hence $f_1(\lambda) \equiv 0 \equiv f_2(\lambda)$ i.e. $f(\lambda) \equiv 0$ $T \in B(x)$ $f : G \to C$

Hence $T_1 \oplus T_2$ has the single –valued extension property.

Conversely Let us suppose that $T_1 \oplus T_2$ has the single –valued extension property and Let $f_\alpha : G \to X_\alpha (G \subseteq C, open; \alpha = 1,2)$ be analytic function such that $(\lambda I_\alpha - T_\alpha) f_\alpha(\lambda) \equiv 0, (\alpha = 1,2)$ then

$[\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2)](f_1(\lambda) \oplus f_2(\lambda)) = (\lambda I_1 - T_1) f_1(\lambda) \oplus (\lambda I_2 - T_2) f_2(\lambda) \equiv 0$

Therefore $f_1(\lambda) \oplus f_2(\lambda) \equiv 0$

hence $f_1(\lambda) \equiv 0 \equiv f_2(\lambda)$ this shows that $T_\alpha (\alpha = 1,2)$ has the single valued extension property.

Now Let $\lambda_0 \in \sigma_{T_1 \oplus T_2}(x_1 \oplus x_2)$ then there exists a neighborhood D of λ_0 and an analytic functions $f = f_1 \oplus f_2 : D \to X_1 \oplus X_2$

such that $(\lambda I_1 - T_1) f_1(X) \oplus (\lambda I_2 - T_2) f_2(\lambda) = [\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2)] f(\lambda) \equiv x_1 \oplus x_2$

Hence $(\lambda I_1 - T_1) f_1(\lambda) \equiv x_1$ & $(\lambda I_2 - T_2) f_2(\lambda) \equiv x_2$

Therefore, $\lambda \in \rho_{T_1}(x_1) \cap \rho_{T_2}(x_2)$

Theorem (3): Let $T \in B(x)$ and $f : G \to C$ be an analytic function non constant on every component of G. Then $f(T)$ has the single valued extension property if and only if T has this property.

Proof: - suppose that T has the single valued extension property and $f(T)$ has not this property. Then there exists an analytic function $h : D \to X$ (where D is a disk) such that $\left[\mu\mathrm{I} - f(\mathrm{T})\right]h(\mu) = 0 \,\&\, h(\mu)$ ------------------------ (1)

This gives $D \subseteq \sigma(f(\mathrm{T})) = f(\sigma(\mathrm{T}))$

Hence for or fixed $\mu \in D$ the equation $\mu - f(\lambda)$ ------- (2)

has a finite number of solutions in $\sigma(T)$. The solutions of equation (2) are of order >1, are also solution of the equation $f'(\lambda) = 0$ ----------------------- (3)

This equation has a finite number of solutions in $\sigma(\mathrm{T})$. Let $\lambda_1^0..........\lambda_k^0$ be these solutions. If we consider a disk $\mathrm{D}_1 \subseteq D\left\{f(\lambda_1^0)............f(\lambda_k^0)\right\}$. Equation (2) has for every $\mu \in \mathrm{D}_1$ only solution of order $=1$. Let $\lambda_1(\mu),...........\lambda_n(\mu)$ be these solutions.

By Rouche Theorem there exists a disk $\mathrm{D}_2 \subseteq D_1$ such that equation (2) has the same number n of solutions $\lambda_1(\mu),............\lambda_n(\mu)$ is G_0 for every $\mu \in \mathrm{D}_2$. Where $\mathrm{G}_0 \subseteq G$ is suitable neighborhood of $\sigma(\mathrm{T})$. These functions are analytic in $\mu \in \mathrm{D}_2$. We can write

$$\mu - f(\lambda) = \left[\lambda - \lambda_1(\mu)\right]\left[\lambda - \lambda_2(\mu)\right]..............\left[\lambda - \lambda_m(\mu)\right]g_\mu(\lambda)\left[\lambda \in G_0, \mu \in D_2\right] \quad .... (4)$$

Where g_μ is an analytic function in λ (and μ) such that $g_\mu(\lambda) \neq 0\left[\lambda \in G_0, \mu \in D_2\right]$.

Therefore $g_\mu(T)$ makes sense that $\sigma(g_\mu(T)) = g_\mu(\sigma(T)) \neq 0$

hence $g_\mu(T)^{-1}$ exists and belongs to $\mathrm{B}(X)$.

From equation (4) it follows by Dun ford's functional calculus

$$\mu\mathrm{I} - f(\mathrm{T}) = \left[\mathrm{T} - \lambda_1(\mu)\mathrm{I}\right]\left[\mathrm{T} - \lambda_2(\mu)\mathrm{I}\right].......\left[\mathrm{T} - \lambda_n(\mu)\right]g_\mu(\mathrm{T})$$

Hence

$$\left[\mathrm{T} - \lambda_1(\mu)\mathrm{I}\right]\left[\mathrm{T} - \lambda_2(\mu)\mathrm{I}\right]........\left[\mathrm{T} - \lambda_n(\mu)\mathrm{I}\right]h(\mu) = g_\mu(\mathrm{T})^{-1}\left[\mu\mathrm{I} - f(\mathrm{T})\right]h(\mu) = 0 \quad .... (5)$$

We observe that all the solutions $\lambda_1(\mu).........\lambda_n(\mu)$ are non-constant. Indeed if there exists a $\lambda_{1_0}(\mu) = \lambda_{1_0}$ then we should have $\mu = f(\lambda_{1_0}(\mu)) \equiv f(\lambda_{1_0})$

Which is not possible (μ being non-constant there exists a $\mu_1^0 \in D$ such that $\lambda_1'(\mu_1^0) \neq 0$)

Therefore there is a disk $D_1^0 = \left\{\lambda \in C : \left|\lambda - \lambda_1(\mu_1^0)\right| < r_1\right\}$ in which λ^{-1} exists.

We have

$$(\mathrm{T} - \lambda\mathrm{I})\left[\mathrm{T} - \lambda_2(\lambda^{-1}(\lambda)\mathrm{I})\right]........\left[\mathrm{T} - \lambda_n(\lambda_1^{-1}(\lambda))\mathrm{I}\right]h(\lambda_1^{-1}(\lambda)) = 0 \quad .... \quad (6)$$

Since T has the single valued extension property and the function

$$\lambda \to \left[\mathrm{T} - \lambda_2(\lambda_1^{-1}(\lambda))\mathrm{I}\right].........\left[\mathrm{T} - \lambda_n(\lambda^{-1}(\lambda))\mathrm{I}\right]n(\lambda_1^{-1}(\lambda))D_1^0 \to X$$ is analytic.

We find that the identity (6) is of the same kind as (5) except for the first factor . Applying to

$\lambda_2^0\lambda_1^{-1}$ the above argument , We have

$\left[\mathrm{T} - \lambda_3(\lambda_2^{-1}(\lambda))\mathrm{I}\right]............\left[\mathrm{T} - \lambda_n(\lambda_2^{-1}(\lambda))\mathrm{I}\right]h(\lambda_2^{-1}(\lambda)) = 0$ on a disk $D \subseteq D_2^0$

If we repeat this procedure we come to the identity $h(\lambda_n^{-1}(\lambda)) = 0$ on a disk D_n^0, hence $h(\mu) = 0$

On the domain $\lambda_n^{-1}(D_n^0)$ therefore $h(\mu) = 0$ on D which is impossible.

Conversely, suppose that f has the single valued extension property and T has not this property. Then there exists an analytic X-valued function defined on a disk D such that $(\lambda I - T)x(\lambda) = 0 \,\&\, x(\lambda) \neq 0$ …………………….. (7)

It follows that $D \subseteq \sigma(T)$ hence $D \subseteq G$: thus f is analytic on D. We have $f(\lambda) - f(\xi) = (\lambda - \xi) g_k(\xi),\ for\,\lambda \in D, \xi \in G$ where γ is also analytic in G.

By DUNFORD'S functional calculus, we get

$f(\lambda)T - f(T) = (\lambda I - T) g_\mu(T)$ ……………… (8)

From equation (7) & (8) , We get $\left[f(\lambda)I - f(T) \right] x(\lambda) = 0$

Since f is not constant on D, We can choose a $\lambda_0 \in D$ such that $f^{/}(\lambda_0) \neq 0$.

Then for $D_0 = \{ \lambda \in C : |\lambda - \lambda_0| < r \}$ With r small enough f^{-1} exists on $f(D_0)$.

So if we put $\gamma(\mu) = x(f^{-1}(\mu)), f(D_0)$.

We get $[\mu I - f(T)]\gamma(\mu) = 0$ Hence $\gamma(\mu) = 0$, from this identity it follows that $x(\lambda) = 0$ on D_0, hence also on D .

Theorem (4): A spectral maximal space of $T \in B(X)$ is ultra-invariant to T i.e. invariant to any operator A commuting with T .

Proof: Let γ be a spectral maximal space of T and for a fixed $\lambda \in \rho(A)$. Let us put $\gamma_\lambda = R(\lambda, A)\gamma$ clearly γ_λ is a closed linear subspace of X . As T commutes with $R(\lambda, A)$ since $TA = AT \,\&\, T\gamma \subseteq \gamma$ it follows that $T\gamma_1 \subseteq \gamma_1$ we remarks also that

$$T/\gamma_\lambda = R(\lambda, A)\left(T/\gamma \right)(\lambda I - A)\, 1/\lambda$$

$$= \left[(\lambda I - A)/\gamma_\lambda \right]^{-1} \left(T/\gamma \right)(\lambda I - A)\, 1/\lambda$$

Hence $\sigma\left(T/\gamma_\lambda \right) = \sigma\left(T/\gamma \right)$, γ being a spectral space of T ,

We have $\gamma_\lambda \subseteq \gamma$ so that $R(\lambda, A)_x \in \gamma, \forall x \in \gamma$.

By Dunford's functional calculus, We have $A = \frac{1}{2\pi i}\int_\Gamma \lambda R(\lambda, A) d\lambda$, where Γ is a system of rectifiable Jordan curves surrounding $\sigma(A)$. Hence $A_x = \frac{1}{2\pi i}\int_\Gamma \lambda R(\lambda, A) x d\lambda, (x \in \gamma)$,

Thus A_x being the limits of sums of the form $\frac{1}{2\pi i}\sum_{k=1}^{n} \lambda_k (\lambda_{k+1} - \lambda_k) R(\lambda_k, A) x \,\&\, \gamma$ being a closed linear sub-space. It follows that $A_x \in \gamma$.

Corollary: For all spectral maximal space γ of T, we have $\sigma\left(\mathrm{T}/\gamma\right) \subseteq \sigma(\mathrm{T})$

Proof: Using the same argument $(for\,\mathrm{A}=\mathrm{T})$ as in the preceding proof, we obtain $R(\lambda,\mathrm{T})\gamma \subseteq \gamma \; for\,\lambda \in \rho(\mathrm{T}); consequently R\left(\lambda, \mathrm{T}/\gamma\right) = R(\lambda,\mathrm{T})/\gamma$

for every $\lambda \in \rho(\mathrm{T}) therefore \rho(\mathrm{T}) \subseteq \rho\left(\mathrm{T}/\gamma\right)$

Corollary: Let γ_1 and γ_2 be two spectral maximal spaces of T then the following two assertions are equivalent $\gamma_1 \subseteq \gamma_2, \sigma\left(\mathrm{T}/\gamma_1\right) \subseteq \sigma\left(\mathrm{T}/\gamma_2\right)$

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