Some generalized results on graph theory in statistical physics

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Abstract :

In this paper we consider that the best way introducing this connection is through generalization of the physical model and the graphical model .We will be walking for some time where the connections between graph theory and statistical physics lead us.

Keywords : Graph, partition function, Ising problem, Tutte polynomial, knot diagram.

Introduction:

The purpose of this note is to describe in passing some beautiful basic concepts internal acing statistical physics, graph theory and Statistical theory.

The connection between statistical physics and graph theory are extensive and have a long history . A survey on these connections was published already in 1971 by Essam it main deals with the Ising model. In the Ising model we consider a set of particles, which can have two sates .The state of the i^{th} particle is described by the variable σ ,which takes one of the two values +1 or -1.The connection with graph theory comes from the calculation of the partition function of model .

The Ising model was introduced in 1920 by Wilhelm Lentz as a model for ferromagnetism, and later studied by Ernst Ising. Ising could only solve it in one dimension; it took until 1944 before it was solved in the 2-dimensional case, without an external magnetic field, by Lars Onsager. The Ising model is perhaps the most studied model in statistical physics. Between 1969 and 1996 thousands of publications on this model have appeared. The project started under the claim that the mathematical language of graph theory could be used to advance the study of spin models used in statistical physics. Our approach is to find scalable relations for the coefficients of the Ising partition function for graph families of interest in statistical physics.

There are also some previously unconsidered questions, inspired by the use of the model as an invariant for graphs, which are of interest from both a mathematical and physical point of view. Other work focuses on the Potts model and its connection to coloring of graphs.

In this model we consider a graph G=(V,E) with each node of which we associate a spin. The spin can have one of q values .The basic physical principal of the model is the energy between two intersecting spins is set to zero for identical spins and it is equal to constant if they are not. In this case the critical singularities in thermodynamics functions are different from those obtained by using the Ising model.

Euler’s Theorem Perhaps the first theorem of graph theory is the Euler’s theorem, and it is also about walking.

Theorem : A graph G = (V,E) has a closed walk containing each edge exactly once if and only if it is connected and each vertex has an even number of edges incident with it.

This theorem has an easy proof. Let us call a set A of edges *even* if each vertex of V is incident with an even number of edges of A. Connectivity and evenness are clearly necessary conditions for the existence of such a closed walk.

In the simplest formulation of the Potts model with q states $\{1,2,3,....q\}$, the Hamiltonian graphical the system can have any two following forms:

$$H_{G_1}(\omega) = -J \sum_{(i,j)\in E} \delta(\sigma_i, \sigma_j) --------(1)$$

$$H_{G_2}(\omega) = J \sum_{(i,j)\in E} \left[1 - \delta(\sigma_i, \sigma_j)\right] ------(2)$$

Where ω is a configuration of the graph, i.e. an assignment of a spin to each node of G=(V,E). σ_i is the spin at node i and δ is the Kronecker symbol. The physical model with zero external field is a special case q=2,so that the spins are +1 and-1.

The probability $p(\omega,\beta)$ of finding the graph in a particular state ω at a given temperature is obtained by the following result

$$p(\omega,\beta) = \frac{e^{-\beta H_{G_i}(\omega)}}{Z_i(G)} ----------------(3)$$

Where $Z_i(G)$ is partition function for a given Hamiltonian in the Graph Model.

The energy (Hamiltonian) of the state

Let us consider all the different spin configuration for cyclic graph with n=4 . aware that are 4 equivalent configuration for ω_2, ω_4 *and* ω_5 as well 2 equivalent for ω_3 . The Hamiltonian $H_{G_1}(\omega)$ for these configuration are

$H_1(\omega_1) = -4J : H_1(\omega_2) = -2J : H_1(\omega_3) = 0; H_1(\omega_4) = -2J;$

Then the partition of this graph is given by

$$Z_i(G) = 12e^{(2\beta J)} + 2e^{(4\beta J)+2} ---------(4)$$

Edge detecting Graphs in Stastical Models :

Lets us define the Tutte polynomial , we define the graph operations. The deletion of an edge ‘e’ in the graphs G ,representing by G-e ,consists of removing the corresponding edge without changing the rest of the graph, i,e .the end nodes of the edge remain in the graph .The other operation is the edge contraction denoted by G/e, which consists in glug together the two end nodes of the edge ‘e’ and then removing ‘e’. Both operations, edge deletion and contraction, are commutative, and the operation G-S and G/S, where S is a subset of edges, are well defined. We notice here that the graphs created by these transformations are no longer simple graphs, they are pseudo graphs which may contain self-looped and multi-looped edges.

We also see some types are a bridge is an edge whose removal disconnects the graph. A self loop is an edge having the two end points incident at the same node.

Then the Tutte Polynomial T(G;x,y) is defined as follows:

(1). T(G;x,y)=T(G-e;x,y)+T(G/e:x,y)

(2).T(G:x,y)=$x^i y^j$

Where i and j represent the number of bridges and self –loops in the subgraphs.

Coloring of a Graph G in Statistical model

Let us consider a proper coloring of a graph G, which is an assignment of color to each node of G such that any two adjacent nodes have different colors. The chromatic polynomial $\chi(G;q)$ of the graph G is the number of ways in which " q" colors can be assigned to nodes of G such that no two adjacent nodes have the same color.

The following are two interesting characteristics of the chromatic polynomial.

$$\lambda(G:q)=\lambda(G-e;q)-\lambda(G/e;q)$$

$$\lambda(G:q)=q^{n} \ for\ the\ trival\ graph\ on\ n\ nodes$$

Thus, the chromatic polynomial fulfill the same contraction/deletion rules as the Tutte polynomial .

Conclusion :

In this paper we will cover some of the most important areas of applications of graph theory in physics. These include condensed matter physics, statistical physics, Thus graph theory and network theory have helped to broaden the horizons of physics to embrace the study of new complex systems. We hope this work motivates the reader to find more about the connections between graph theory and physics.

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