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ON THE FUZZY METRIC PLACES

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ABSTRACT----*Zadeh [Zadeh , 1965] introduced the concepts of fuzzy sets in1965 , and in the next decade Kramosil and Michalek [Kramosil & Michalek ,1975] introduced the concept of fuzzy metric space with the help of continuous t –norms in1975 which opened an avenue for further development of analysis in such spaces which have very important applications in quantumphysics particularly in connections with both string and $\in^{(\infty)}$ theory which were given and studied by EI Naschie [El Naschie ,1998] . George and Veeramani [George &Veeramani ,1994,1997] modified the concept of fuzzyPmetric space introduced by Kramosil and Michalek also with the help of continuous t – norms.*

In this search we will define in different way the fuzzy metric space by given definitions about the fuzzy families, the fuzzy field, the fuzzy space, and other concepts based on that every real number r is replaced by fuzzy number $\bar{r}$ (either triangular fuzzy number or singleton fuzzy set). For more details see [Kramosil &Michalek, 1975] [Erceg, 1979], [Grabiec, 1988], [Kaleva & Seikkala, 1984].

KeyWord: **Fuzzy Metric Space, Triangular Fuzzy number, Operations of Fuzzy numbers, Fuzzy Pseudo-Metric, Fuzzy mapping.**

1. Some Definitions and Concepts about The Fuzzy Set

Definition 1.1. [Zadeh, 1965]

If X is a collection of objects denoted generically by x , then a fuzzy set A in X is a set of order pairs:-

$$\bar{A} = \left\{ \left(x, \bar{A}(x) \right) : x \in X \right\}$$

$A(x)$ is called the membership function or grade of membership of x in A that maps X to the unite interval [0, 1].

Definition 1.2. [Zadeh, 1965]

The standard intersection of fuzzy sets *A* and *B* is defined as

$$\left(\bar{A} \cap \bar{B}\right)(x) = \min\left\{\bar{A}(x), \bar{B}(x)\right\} \quad \text{for all}$$
$$= \bar{A}(x) \wedge \bar{B}(x)$$
$$x \in X .$$

Definition 1.3. [Zadeh, 1965]

The standard union of fuzzy sets *A* and *B* is defined as

$$\left(\overline{A}\cup\overline{B}\right)(x)=\max\left\{\overline{A}(x),\overline{B}(x)\right\}$$
$$=\overline{A}(x)\vee\overline{B}(x)$$
For all $x\in X$.

Definition 1.4. [Zadeh, 1965]
The standard complement of a fuzzy set A is defined

as $\left(\neg\overline{A}\right)(x)=1-\overline{A}(x)$.

Definition 1.5. [Zadeh, 1965]
Let A be a fuzzy set of X , the support of A , denoted S A is the crisp set of X whose elements all have non zero membership grades in A , that is

$$S\left(\overline{A}\right)=\left\{x\in X:\overline{A}(x)>0\right\}.$$

Definition 1.6. [Zadeh, 1965]
(α-cut) An α- level set of a fuzzy set $\overline{A}$ of X is a non fuzzy (crisp) set denoted by $A\left[\alpha\right]$, such that

$$\overline{A}[\alpha]=\begin{cases}\left\{x\in X:\overline{A}(x)\geq\alpha\right\}, & if\alpha>0\\ cl\left(S\left(\overline{A}\right)\right) & , if\alpha=0\end{cases}$$

Where $cl(S(A))$ denotes closure of the support of A .

Theorem 1.7. [Chandra & Bector, 2005]

Let $\overline{A}$ be a fuzzy set in X with the membership function $A(x)$. Let $[\alpha]\ A$ be the α -cuts of A and $\chi_{A[\alpha]}(x)$ be the characteristic function of the crisp set $[\alpha]\ A$ for all $\alpha\in[0,1]$.

.

Then $\overline{A}(x)=\sup_{\alpha\in[0,1]}\left(\alpha\wedge\chi_{\overline{A}[\alpha]}(x)\right),\ x\in X$.

Given a fuzzy set A in X, one consider a special fuzzy set denoted $\alpha A[\alpha]$ for $\alpha\in[0,1]$ whose membership function is defined as

$$\overline{A}_{\alpha\overline{A}[\alpha]}(x)=\left(\alpha\wedge\chi_{\overline{A}[\alpha]}(x)\right),\ x\in X$$

is called the level set of A. Then the above theorem states that the fuzzy set A can be expressed in the form

$$\overline{A}=\bigcup_{\alpha\in\Lambda_{\overline{A}}}\left(\alpha\overline{A}[\alpha]\right)$$

Where denotes the standard fuzzy union. This result is called the resolution principle of fuzzy sets. The essence of resolution principle is that a fuzzy set A can be decomposed in to fuzzy sets $\alpha A[\alpha],\alpha\in[0\ ,\ 1]$. Looking from a different angle, it tells that a fuzzy set A in X can be retrieved as a union of its $\alpha A[\alpha]$ sets $\alpha\in[0\ ,\ 1]$. This is called the representation theorem of fuzzy sets. Thus the resolution principle and the representation theorem are the two sides of the same coin as both of them essentially tell a fuzzy set A in X can always be expressed in terms of its α-cuts without explicitly resorting to its membership function $A(x)$.

Definition 1.8. [Chandra & Bector, 2005]

A fuzzy set $\overline{A}$ of a classical set X is called normal, if there exists an $x \in X$, such that $A(x)= 1$. Otherwise A is subnormal.

Definition 1.9.[Zadeh , 1965]

A fuzzy set $\overline{A}$ of $\overline{X}$ is called convex, if $A[\alpha]$ is a convex subset of X, for all $\alpha \in [0,1]$. That is, for any $x , y \in A[\alpha]$, and for any $\lambda \in [0,1]$ then λ (λ) [α] $x + 1- y \in A$.

Definition 1.10. [Bushera, 2006]
A fuzzy set A whose $S(A)$contains a single point $x \in X$, with ( ) 1 A x =, is referred to as a singleton fuzzy set.

Definition 1.11. [Bushera, 2006]
The empty fuzzy set of X is defined as $\Phi = \{(x,0): \forall x \in X\}$

Definition 1.12. [Bushera, 2006]
The largest fuzzy set in X is defined as $I_X = \{(x,1) : \forall x \in X\}$

Definition 1.13. [Bushera, 2006]
The concept of continuity is same as in other functions, that say, a function f is continuous at some number c if

$f(x)\ f(c)\ _{x\ c} = \to \lim$ For all x in range of f, that require existing $f(c)$ and $f(x)\ _{x \to c} \lim$. In fuzzy set theory the condition will be $A(x)\ A(c)\ _{x\ c}\_\_ \lim = \to$ with x and $\_c \in A$.

Definition 1.14. [Zadeh, 1965]

A fuzzy set $\bar{A}$ is said to be a bounded fuzzy set , if itα -cuts $A[\alpha]$ are (crisp) bounded sets , for all$\alpha \in [0,1]$.

Definition 1.15. [Zadeh, 1965]

A fuzzy number $\bar{A}$ is a fuzzy set of the real line with a normal, (fuzzy) convex, and continuous membership function of bounded support.

Example 1.16. [Zadeh, 1965]
The following fuzzy set is fuzzy number approximately
"5"= {(3,0.2), (4,0.6),(5,1.0),(6,0.7),(7,0.1)}.

Proposition 1.17. [Abdull Hameed, 2008]

Let $\bar{A}$ be a fuzzy number, then $\bar{A}[\alpha]$ is a closed, convex, and compact subset of R , for all $\alpha \in [0, 1]$.
right hand side function which monotone decreasing and continuous.

Remark 1.18.

We shall use the notation $[\alpha]\ [\ (\alpha)\ (\alpha)]\ _{1\ 2} A = a\ , a$, where $[\alpha]\ A$ is an α -cut of the fuzzy number A , and $a : [0,1] \to R\ _1 , (\alpha)\ [\alpha]\ _1 a = \min A$, is left hand side function which monotone, increasing and continuous $a : [0,1] \to R\ _2 , (\alpha)\ [\alpha]\ _2 a = \max A$ is right hand side function which monotone decreasing and continuous.

Proposition 1.19. [Abdull Hameed, 2008]

If $\alpha \le \beta$, then $\bar{A}[\alpha] \supset \bar{A}[\beta]$.

Proposition 1.20. [Abdull Hameed, 2008]

The support of a fuzzy number is an open interval $(a_1(0), a_2(0))$.

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Definition 1.21. [Zimmerman, 1995]

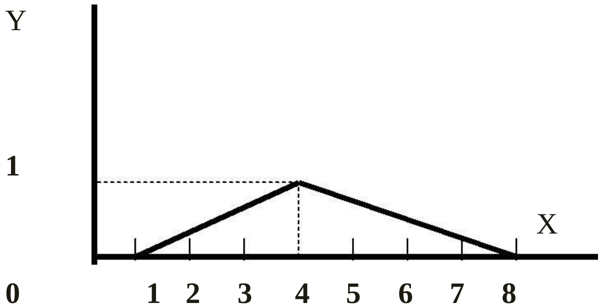
Let A be a fuzzy number. If $S(A) = \{x\}$ then A is called a fuzzy point and we use the notion $\bar{A} = \bar{x}$. Let $\bar{A} = \bar{x}$ be a fuzzy point, it is easy to see that $\bar{A}[\alpha] = [x, x] = \{x\}, \forall \alpha \in [0,1]$.

Here it may be remarked that the reason for $\bar{A} * \bar{B}$ to be a fuzzy number, and not just a general fuzzy set, is that $\bar{A}$ and $\bar{B}$ being fuzzy numbers, the sets $\bar{A}[\alpha]$, $\bar{B}[\alpha]$, $(\bar{A} * \bar{B})[\alpha]$, are all closed intervals for all $\alpha \in [0,1]$.

$\bar{A}[\alpha] = [(a_2 - a_1)\alpha + a_1, (a_2 - a_3)\alpha + a_3]$ for all $\alpha \in [0,1]$.

Example 1.23.[Zadeh , 1965]

$\bar{A} = (1, 4, 8)$ is triangular fuzzy number , where



triangular fuzzy number.

In particular

$$\bar{A}[\alpha](+)\bar{B}[\alpha] = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)]$$

$$\bar{A}[\alpha](-)\bar{B}[\alpha] = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)]$$

Further, for fuzzy numbers $\bar{A}$ and $\bar{B}$ in R^+

$$\bar{A}[\alpha](\cdot)\bar{B}[\alpha] = \begin{bmatrix} \min(a_1(\alpha)\cdot b_1(\alpha), a_1(\alpha)\cdot b_2(\alpha), a_2(\alpha)\cdot b_1(\alpha), a_2(\alpha)\cdot b_2(\alpha)), \\ \max(a_1(\alpha)\cdot b_1(\alpha), a_1(\alpha)\cdot b_2(\alpha), a_2(\alpha)\cdot b_1(\alpha), a_2(\alpha)\cdot b_2(\alpha)) \end{bmatrix}$$

$$\bar{A}[\alpha](:)\bar{B}[\alpha] = \left[\min\left(\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)} \right), \left(\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)} \right) \right]$$

$0 \notin [b_1(\alpha), b_2(\alpha)]$

Example 2.2. [George, 1995]

2. Arithmetic Operations on Fuzzy Numbers

We will define the arithmetic operations on fuzzy numbers based on resolution principle (α -cuts).

Definition 2.1. [George, 1995]

Consider two triangular fuzzy numbers $\bar{A}$ and $\bar{B}$ defined as

Let $\bar{A}$ and $\bar{B}$ be two fuzzy numbers and $\bar{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$, $\bar{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$ be α-cuts ,$\alpha \in [0,1]$ of $\bar{A}$ and $\bar{B}$ respectively . Then the operation (denoted anyof the arithmetic

$$\bar{A}(x) = \begin{cases} 0 & \text{for } x \le -1,\ x \ge 3 \\ \dfrac{(x+1)}{2} & \text{for } -1 < x \le 1, \\ \dfrac{(3-x)}{2} & \text{for } 1 < x < 3 \end{cases}$$

(idempotence)

iv) $MIN\left(\bar{A}, MAX\left(\bar{A}, \bar{B}\right)\right) = \bar{A}$ (absorption)

$MAX\left(\bar{A}, MIN\left(\bar{A}, \bar{B}\right)\right) = \bar{A}$

$$\bar{B}(x) = \begin{cases} 0 & \text{for } x \le 1,\ x \ge 5 \\ \dfrac{(x-1)}{2} & \text{for } 1 < x \le 3 \\ \dfrac{(5-x)}{2} & \text{for } 3 < x < 5 \end{cases}$$

v) $MIN\left(\bar{A}, MAX\left(\bar{B}, \bar{C}\right)\right) = MAX\left(MIN\left(\bar{A}, \bar{B}\right), MIN\left(\bar{A}, \bar{C}\right)\right)$

$MAX\left(\bar{A}, MIN\left(\bar{B}, \bar{C}\right)\right) = MIN\left(MAX\left(\bar{A}, \bar{B}\right), MAX\left(\bar{A}, \bar{C}\right)\right)$

Their α-cuts are

Now

$$\left(\bar{A}+\bar{B}\right)[\alpha] = [4\alpha, 8 - 4\alpha],\ \alpha \in [0,1]$$

The resulting fuzzy number is then

(distributivity).

The triple $(\Re, MIN, MAX)$ is called lattice of fuzzy numbers. The triple $(\Re, MIN, MAX)$ can be

$$\left(\bar{A}+\bar{B}\right)(x) = \begin{cases} 0 & \text{for } x \le 0,\ x \ge 8 \\ \dfrac{x}{4} & \text{for } 0 < x \le 4 \\ \dfrac{(8-x)}{4} & \text{for } 4 < x < 8 \end{cases}$$

expressed as the pair $(\Re, \le)$, where $\le$ is a partial ordering defined as:

$\bar{A} \le \bar{B}$ iff $MIN\left(\bar{A}, \bar{B}\right) = \bar{A}$

or, alternatively

3. Lattice of Fuzzy Numbers [George, 1995]

Let $\Re$ denote the set of all fuzzy numbers. Then the operations MIN and $MIAX$ are functions of the form $\Re \times \Re \to \Re$ such that:-

i) $MIN\left(\bar{A}, \bar{B}\right) = MIN\left(\bar{B}, \bar{A}\right)$ (commutat

$MAX\left(\bar{A}, \bar{B}\right) = MAX\left(\bar{B}, \bar{A}\right)$

ivity)

ii) $MIN\left[MIN\left(\bar{A}, \bar{B}\right), \bar{C}\right] = MIN\left[\bar{A}, MIN\left(\bar{B}, \bar{C}\right)\right]$

$MAX\left[MAX\left(\bar{A}, \bar{B}\right), \bar{C}\right] = MAX\left[\bar{A}, MAX\left(\bar{B}, \bar{C}\right)\right]$

(associativity).

$\bar{A} \le \bar{B}$ iff $MAX\left(\bar{A}, \bar{B}\right) = \bar{B}$ for any $\bar{A}, \bar{B} \in \Re$.

Now, this partial ordering can be defined in terms of the relevant α-cuts:-

$\bar{A} \le \bar{B}$ iff $MIN\left(\bar{A}[\alpha], \bar{B}[\alpha]\right) = \bar{A}[\alpha]$, $\bar{A} \le \bar{B}$ iff $MAX\left(\bar{A}[\alpha], \bar{B}[\alpha]\right) = \bar{B}[\alpha]$

for any

$\bar{A}, \bar{B} \in \Re$ and $\alpha \in [0,1]$,where $\bar{A}[\alpha]$ and $\bar{B}[\alpha]$ are closed intervals, then

$MIN\left(\bar{A}[\alpha], \bar{B}[\alpha]\right) = [MIN(a_1, b_1), MIN(a_2, b_2)], MAX\left(\bar{A}[\alpha], \bar{B}[\alpha]\right) = [MAX(a_1, b_1), MAX(a_2, b_2)]$

If we define the partial ordering of closed intervals, that is

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$[a_1, a_2] \leq [b_1, b_2]$ iff $a_1 \leq b_1$, $a_2 \leq b_2$

Then for any $\bar{A}, \bar{B} \in \Re$,we have $\bar{A} \leq \bar{B}$ iff for all $\alpha \in [0,1]$. For example, we have in example 2.2 that $\bar{A} \leq \bar{B}$ since $A[\alpha] \leq B[\alpha]$ for all $\alpha \in [0,1]$.

4. Fuzzy Families

Definition 4.1.

The set of natural numbers is $N = \{1,2,3,...\}$.

The set of integer numbers is $Z = \{..., -3,-2,-1,0,1,2,3,...\}$.

The set of rational numbers is $Q = \{a/b : a, b \in Z , b \neq 0\}$.

In other words, every terminating or recurring decimal is a rational number.

That is, every non terminating and non recurring decimal is an irrational number. The set of all irrational numbers is denoted by Q'.The set $R = Q \cup Q'$ is called the set of real numbers. **Note that:-** If we substitute every real number $r \in R$ by a fuzzy number $\bar{r}$, such that:-

If $r \in Z$, we replace r by a singleton fuzzy set $\bar{r}$. The set of all fuzzy numbers $\bar{r}$, $r \in Z$, will be called the family of fuzzy integer numbers , and denoted by $\bar{\bar{Z}}$, where $\bar{\bar{Z}} = \{..., -\bar{2}, -\bar{1}, \bar{0}, \bar{1}, \bar{2},...\}$. The family of all fuzzy natural numbers will be $\bar{\bar{N}} = \{\bar{1}, \bar{2}, \bar{3},...\}$.

ii) Z_-

iii) Q_0

iii) Q_+

Because dense of rational and irrational numbers, we replace every rational r and irrational numbers r' by a triangular fuzzy number $\bar{r} = (r_1, r_2, r_3)$, it α-cut's $\bar{r}[\alpha] = [r_1(\alpha), r_2(\alpha)], \alpha \in [0,1]$, and $\bar{r}' = (r'_1, r'_2, r'_3)$, it α-cut's $\bar{r}'[\alpha] = [r'_1(\alpha), r'_2(\alpha)]$, $\alpha \in [0,1]$ respectively.

The set of all fuzzy numbers $\bar{r}$, $r \in Q$ and the set of all fuzzy numbers $\bar{r}'$, $r' \in Q'$, will be called the family of fuzzy rational numbers and the family of fuzzy irrational numbers which denoted by $\bar{\bar{Q}}$ and $\bar{\bar{Q}}'$ respectively.

The set $\bar{\bar{R}} = \bar{\bar{Q}} \cup \bar{\bar{Q}}'$ will be called the family of fuzzy real numbers.

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Remark 4.2.

The fuzzy numbers mean here either triangular fuzzy number or singleton fuzzy set.

Definition 4.3.

From definition 1, we can define the following: $i)$ $\overline{\overline{Z}}_0$ the family of all non- fuzzy zero fuzzy integer numbers .That is, for all ,

then $\bar{r} \neq \bar{0}$.

the family of all negative fuzzy integer numbers .That is, for all $\bar{r} \in \overline{\overline{Z}}_-$, then $\bar{r} < \bar{0}$.

the family of all non- fuzzy zero fuzzy rational numbers .That is, for all $\bar{r} \in \overline{\overline{Q}}_0$, then $\bar{r} \neq \bar{0}$.

the family of all positive fuzzy rational numbers .That is, for all , then $\bar{r} > \bar{0}$.

$iv)$ $\overline{\overline{Q}}_-$ the family of all negative fuzzy rational numbers .That is, for all $\bar{r} \in \overline{\overline{Q}}_-$, then $\bar{r} < \bar{0}$.

$v)$ $\overline{\overline{R}}_0$ the family of all non-fuzzy zero fuzzy real numbers.That is , for all $\bar{r} \in \overline{\overline{R}}_0$, then $\bar{r} \neq \bar{0}$.

$vi)$ $\overline{\overline{R}}_+$ the family of all positive fuzzy real numbers .That is, for all $\bar{r} \in \overline{\overline{R}}_+$, then $\bar{r} > \bar{0}$.

$vii)$ $\overline{\overline{R}}_-$ the family of all negative fuzzy real numbers .That is, for all $\bar{r} \in \overline{\overline{R}}_-$, then $\bar{r} < \bar{0}$.

vii) $\overline{\overline{N}}_{\bar{k}}$ the family of all fuzzy natural numbers which are less or equal to $\bar{k}$, where $\bar{k}$ is a positive fuzzy integer .Thus $\overline{\overline{N}}_{\bar{k}} = \{\bar{1}, \bar{2}, ..., \bar{k}\}$.

Definition 4.4. [Sharma, 1977]
The sets of the forms $\{x \in R : a < x < b\}, \{x \in R : a \le x \le b\}, \{x \in R : a \le x < b\}, \{x \in R : a < x \le b\}$ are called open interval, closed interval, right half open interval and left half open interval, and denoted by respectively. The sets of
the forms $\{x \in R : x > a\}, \{x \in R : x < a\}, \{x \in R : x \ge a\}, \{x \in R : x \le a\}$ are called rays and denoted by $(a, \infty), (-\infty, a), [a, \infty), (-\infty, a]$ respectively. The first two rays are

it. x

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respe ctively. The last two families are called closed

fuzzy rays and will be denoted by $\left[a, \bar{\infty}\right), \left(\bar{-\infty}, \bar{a}\right]$ respectively.

Definition 4.5. [Bhattacharya *et al.*,1989]

A set S is a collection of objects (or elements). If S is a set, and x is an element of the set S, we say that x belongs to S, and we write $x \in S$. If x doesn't belong to S ,we write $x \notin S$.

Note:- Since the real numbers is essential to every set S , and the elements x of S is one forms of real numbers. Hence, if we have the family of fuzzy real numbers $\bar{\bar{R}}$, the fuzzy number $\bar{x}$ will become one forms of fuzzy real numbers, and $\bar{\bar{S}}$ will be a family of fuzzy numbers x .

called open rays, and the last two rays are called closed rays.

Note:- Suppose we have the family of fuzzy real numbers .The families of the forms

$\left\{\bar{x}\in\bar{\bar{R}} :\bar{a}< \bar{x}<\bar{b}\right\}, \left\{\bar{x}\in\bar{\bar{R}} : \bar{a} \le \bar{x}\le \bar{b}\right\},\left\{\bar{x}\in\bar{\bar{R}}: \bar{a} \le \bar{x} <\bar{b}\right\},\left\{\bar{x}\in\bar{\bar{R}}: \bar{a} < \bar{x} \le\bar{b}\right\}$

Definition 4.6. [Bhattacharya *et al.*, 1989]

Let x, y be elements of a set S . The set $\{\{x\}, \{x, y\}\}$ is called an order pair and is denoted by (x, y). x is called the first component (or coordinate) , and y is the second component (or coordinate). will be called open fuzzy interval , closed fuzzy interval , right half open fuzzy interval and left half open fuzzy interval , and will be denoted by $(\bar{a}, \bar{b}), [\bar{a}, \bar{b}], (\bar{a}, \bar{b}], [\bar{a}, \bar{b})$ respectively, if the sets at degree α , $\{x_\alpha \in R : a_\alpha < x_\alpha < b_\alpha\}$, $\{x_\alpha \in R : a_\alpha \le x_\alpha \le b_\alpha\}$, $\{x_\alpha \in R : a_\alpha \le x_\alpha < b_\alpha\}$, $\{x_\alpha \in R : a_\alpha < x_\alpha \le b_\alpha\}$ are open interval ,closed interval , right half open interval and left half open interval respectively , for all $\alpha \in [0,1]$.

The families of the forms $\{\bar{x} \in \bar{\bar{R}} : \bar{x} > \bar{a}\}$, $\{\bar{x} \in \bar{\bar{R}} : \bar{x} < \bar{a}\}$, $\{\bar{x} \in \bar{\bar{R}} : \bar{x} \ge \bar{a}\}$, $\{\bar{x} \in \bar{\bar{R}} : \bar{x} \le \bar{a}\}$ will be called the fuzzy rays, if the sets at degree α , $\{x_\alpha \in R : x_\alpha > a_\alpha\}$, $\{x_\alpha \in R : x_\alpha < a_\alpha\}$. are rays , for all $\alpha \in [0,1]$.The first two families will be called open fuzzy rays, and denoted by $(\bar{a}, \bar{\infty}), (-\bar{\infty}, \bar{a})$

Note:- Let $\bar{x}, \bar{y}$ be fuzzy elements of the family of fuzzy numbers $\bar{\bar{S}}$. The family of the families of fuzzy numbers $\{\{\bar{x}\}, \{\bar{x}, \bar{y}\}\}$ will be called an order fuzzy pair, and will be denoted by $(\bar{x}, \bar{y})$, where $(\bar{x}, \bar{y})$ consists of all order pairs at degree α, $(x_\alpha, y_\alpha) \in x[\bar{\alpha}] \times y[\bar{\alpha}] = (\bar{x} \times \bar{y})[\bar{\alpha}]$ α-cut of $\bar{x}$ will be called the first fuzzy component (or fuzzy coordinate), and $\bar{y}$ will be called the second fuzzy component (or fuzzy coordinate).

Definition 4.7. [Bhattacharya *et al.*, 1989]

Let A, B be sets. The set of all order pairs (x, y), $x \in A$, $y \in B$ is called the Cartesian product of A and B, and is denoted by $A \times B$, where $A \times B = \{(x, y): x \in A, y \in B\}$.

Note:- If we have families of fuzzy numbers.

The family of all order fuzzy pairs $(\bar{x}, \bar{y}), \bar{x} \in \bar{\bar{A}}, \bar{y} \in \bar{\bar{B}}$ will be called the fuzzy Cartesian product of $\bar{\bar{A}}$ and $\bar{\bar{B}}$, and will be denoted by $\bar{\bar{A}} \times \bar{\bar{B}}$ such that $\bar{\bar{A}} \times \bar{\bar{B}} = \{(\bar{x}, \bar{y}) : \bar{x} \in \bar{\bar{A}}, \bar{y} \in \bar{\bar{B}}\}$.

Definition 4.8. [Bhattacharya *et al.*, 1989]

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the fuzzy codomain respectively .The fuzzy

$\overline{\text{all}}$ $\overline{\text{fuzzy}}$ images under $f\left[A\right]$.

Thus $\overline{\overline{f}}\left[\overline{\overline{A}}\right] = \left\{ y \in \overline{\overline{B}} : \overline{y} = \right.$

and of the fuzzy mapping f range of f is the family of f, and will be denoted by

$\overline{f}(\overline{x}), \overline{x} \in \overline{\overline{A}}\} = \{\overline{f}(\overline{x}) : \overline{x} \in \overline{\overline{A}}\}$.

Let A, B be sets, and let E be a subsets of .Then Eis called a relation from A to B. If $(x, y) \in E$, then x is said to be in relation E to y, written xEy.

Note that:-Let $\overline{\overline{A}}$, $\overline{\overline{B}}$ be families of fuzzy numbers**,** and let $\overline{\overline{E}}$ be a subset of . Then $\overline{\overline{E}}$ will be called a fuzzy relation from $\overline{A}$ to $\overline{B}$. If $(\overline{x}, \overline{y}) \in \overline{E}$, then $\overline{x}$ will be said in fuzzy relation $\overline{\overline{E}}$ to $\overline{y}$, written $\overline{x}\,\overline{\overline{E}}\,\overline{y}$.

Definition 4.9. [Bhattacharya, *et al.*, 1989]

Let A, B be sets .A relation f from A to B is called a mapping (or a map or a function) from A to B, if for each element x in A, there is exactly one element y in B (called the image of x under f), such that , x is in relation f to y. If f is a mapping from A to B, we write $f : A \to B$ .The sets A and B are called the domain and the codomain of the mapping f respectively. The range of f is the set of all images under f, and is denoted by $f[A]$. Thus

$f[A] = \{y \in B : y = f(x), x \in A\} = \{f(x) : x \in A\}$.

Note:-Let $\overline{\overline{A}}$, $\overline{\overline{B}}$ be families of fuzzy numbers. A fuzzy relation $\overline{\overline{f}}$ will be called a fuzzy mapping (or a fuzzy map or a fuzzy function) from A to , if for each fuzzy element $\overline{x}$ in $\overline{\overline{A}}$, there is exactly one fuzzy element $\overline{y}$ in $\overline{B}$. If $\overline{f}$ is a fuzzy mapping from $\overline{A}$ to $\overline{B}$, we will write $\overline{f} : \overline{A} \to \overline{B}$ .The families $\overline{\overline{A}}$ and $\overline{\overline{B}}$ will be called the fuzzy domain

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Definition 4.10. [Bhattacharya *et al.*,1989]

Let $f : A \to B$ and $g : B \to C$ be mappings. Then the mapping $h : A \to C$ given by $h(x) = g(f(x))$ for all $x \in A$, is called the composite of f followed by g and is denoted by $g \; f$. The mappings f and g are called factors of the composite $h = g \; f$.

Note:- Let $\bar{\bar{f}} : \bar{\bar{A}} \to \bar{\bar{B}}$ and $\bar{\bar{g}} : \bar{\bar{B}} \to \bar{\bar{C}}$ be fuzzy mappings. Thenthefuzzy mapping $\bar{\bar{h}} : \bar{\bar{A}} \to \bar{\bar{C}}$ given by $\bar{h}(\bar{x}) = \bar{g}(\bar{f}(\bar{x}))$ for all $\bar{x} \in \bar{\bar{A}}$ will be called the fuzzy composite of $\bar{\bar{f}}$ followed by $\bar{\bar{g}}$, and will be denoted by $\bar{\bar{g}} \; \bar{\bar{f}}$. The fuzzy mappings $\bar{\bar{f}}$ and $\bar{\bar{g}}$ will be called fuzzy factors of the fuzzy composite $\bar{\bar{h}} = \bar{\bar{g}} \; \bar{\bar{f}}$.

Definition 4.11. [Ruel, *et al.*, 1974]

Complex numbers z can be defined as ordered pairs $z = (x, y)$ of real numbers x and y .Complex numbers of the form $(0, y)$ are called pure imaginary numbers. x and y are called the real and imaginary parts of z respectively, and we write $\operatorname{Re} z = x$, $\operatorname{Im} z = y$. In particular, $(x,0) + (0, y) = (x, y)$,

and $(0,1)(y,0) = (0, y)$, hence $(x, y) = (x,0) + (0,1)(y,0)$.

Any order pair $(x,0)$ is to be identified as the real number x , and so the set of complex numbers includes the real numbers as a subset. Let i denote the pure imaginary number $(0,1)$, we can write (x, y) as $(x, y) = x + iy$ .We note that $i^2 = (0,1)(0,1) = (-1,0) = -1$, so .

Note:- If we have $\overline{\overline{R}}$ the family of fuzzy real numbers .The fuzzy complex numbers $\overline{z}$ will be defined as ordered fuzzy pairs $\overline{z}=(\overline{x},\overline{y})$ of fuzzy numbers $\overline{x}$ and $\overline{y}$. $\overline{x}$ and $\overline{y}$ will be called the fuzzy real and fuzzy imaginary parts of $\overline{z}$ respectively, we will write $f\operatorname{Re}\overline{z}=\overline{x}, f\operatorname{Im}\overline{z}=\overline{y}$. The family of fuzzy complex numbers will be denoted by $\overline{\overline{C}}$.

Fuzzy complex numbers of the form $(\overline{0},\overline{y})$ will be called pure fuzzy imaginary numbers . In particular $(\overline{x},\overline{0})+(\overline{0},\overline{y})=(\overline{x},\overline{y})$, and $(\overline{0},\overline{1})(\overline{y},\overline{0})=(\overline{0},\overline{y})$, hence $(\overline{x},\overline{y})=(\overline{x},\overline{0})+(\overline{0},\overline{1})(\overline{y},\overline{0})$. Because any order fuzzy pair is identified as the fuzzy real number .

Hence the family of fuzzy complex numbers $\overline{\overline{C}}$ includes the family of fuzzy real numbers $\overline{\overline{R}}$ as subset .Let $\overline{i}$ denote the pure fuzzy imaginary number , we can write the fuzzy complex numbers $\overline{x},\overline{y}$ as $\overline{x},\overline{y}$ $=\overline{x}+\overline{i}\,\overline{y}$ .Since $i\in \overline{i}[\alpha]$ α-cut of $\overline{i}$, for all $\alpha\in[0,1]$ is $i^2=-1$, then $i^2\in\overline{i^2}[\alpha]$ of $\overline{i^2}$,

Definition 4.12

From definition 8, we can define C_0 the family of all non- fuzzy zero fuzzy complex numbers. That is,

for all $\alpha\in[0,1]$. Hence $i^2=-1$. Also $i=\sqrt{-1}$.

Note:- If we have $\overline{\overline{R}}$ the family of fuzzy real numbers . The $n-$dimensional fuzzy Euclidean space $\overline{\overline{R}}^n$ will be obtained, if we take the family of all ordered $n-$tuples of fuzzy real numbers, and written $\overline{\overline{R}}^n = \{\bar{x} = (\overline{x_1}, \overline{x_2}, ..., \overline{x_n}) : \overline{x_i} \in \overline{\overline{R}}, i = 1, ..., n\}$, where $\bar{x} = (\overline{x_1}, \overline{x_2}, ..., \overline{x_n})$ consists of all ordered $n-$tuples of real numbers at degree α,

α-cut of it .

Definition 4.14. [Royden, 1966]
A sequence $\langle x_n \rangle$ in a set is a function from the set N to X. The value of the function at $n \in N$, being denoted by x_n.
Note:- If we have the family of the fuzzy numbers $\overline{\overline{X}}$, and the family of the fuzzy natural numbers $\overline{\overline{N}}$. The fuzzy sequence $\langle \overline{x_n} \rangle$ in $\overline{\overline{X}}$ will be a fuzzy function from $\overline{\overline{N}}$ to $\overline{\overline{X}}$. A fuzzy sequence $\langle \overline{x_n} \rangle$ consists of all ordered tuples (sequence) at degree α

$$\langle x_{in,\alpha} \rangle = \langle x_{i1,\alpha}, x_{i2,\alpha}, x_{i3,\alpha}, ... \rangle \in \langle \overline{x_1}[\alpha], \overline{x_2}[\alpha], \overline{x_3}[\alpha], ... \rangle = \langle \langle \overline{x_1}, \overline{x_2}, \overline{x_3}, ... [\alpha] \rangle \rangle$$

,$\alpha-$cut of it, where i is fixed belong to N. The fuzzy value of the fuzzy function at $\bar{n} \in \overline{\overline{N}}$, will be denoted by $\overline{x_n}$.
for all $\bar{z} \in \overline{\overline{C_0}}$, then $\bar{z} \neq \bar{0}$.

Definition 4.13. [Kreyszig, 1978]
The $n-$dimensional Euclidean space R^n is obtained, if we take the set of all ordered $n-$tuples of real numbers, written,
$R^n = \{x = (x_1, x_2, ..., x_n) : x_i \in R, i = 1, ..., n\}$.

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Examples 4.15

1) $\left\langle \overline{-1}^{\,n} \right\rangle = \left\langle \overline{-1}, \overline{1}, \overline{-1}, \overline{1}, \ldots, \overline{-1}^{\,n}, \ldots \right\rangle$

2) $\left\langle \bar{3} \right\rangle = \left\langle \bar{3}, \bar{3}, \ldots, \bar{3}, \ldots \right\rangle$

3) $\left\langle \left(\frac{1}{n+1}, \frac{1}{n}, \frac{1}{n-1} \right) \right\rangle = \left\langle \left(\frac{1}{3}, \frac{1}{2}, 1 \right), \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{2} \right), \left(\frac{1}{5}, \frac{1}{4}, \frac{1}{3} \right), \ldots, \left(\frac{1}{n+1}, \frac{1}{n}, \frac{1}{n-1} \right) \right\rangle$

4) $\left\langle \left(\frac{1}{2^{n+1}}, \frac{1}{2^{n}}, \frac{1}{2^{n-1}} \right) \right\rangle = \left\langle \left(\frac{1}{4}, \frac{1}{2}, 1 \right), \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{2} \right), \left(\frac{1}{16}, \frac{1}{8}, \frac{1}{4} \right), \ldots, \left(\frac{1}{2^{n+1}}, \frac{1}{2^{n}}, \frac{1}{2^{n-1}} \right), \ldots \right\rangle$

Definition 4.16.[Sharma , 1977]

Let $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ be two sequences in a set X. Then is said to be a subsequence of x if there exists a mapping $\varphi : N \to N$, such that :-

$i)$ $y = x \circ \varphi$

$ii)$For each n in N, there exists an m in N such that$\varphi(i) \geq n$ for every $i \geq m$ in N.

Note that:-If we have $\bar{x} = \langle \bar{x}_n \rangle$, and $\bar{y} = \langle \bar{y}_n \rangle$ be two fuzzy sequences in a family of fuzzy numbers $\bar{\bar{X}}$. Then $\bar{\bar{y}}$ will be said a fuzzy subsequence of $\bar{\bar{x}}$, if there exists a fuzzy mapping $\varphi : \bar{\bar{N}} \to \bar{\bar{N}}$, such that :-

$i)$ $\dot{y} = \dot{x} \circ \dot{\varphi}$

$ii)$For each $\bar{n}$ in $\bar{\bar{N}}$, there exists an $\bar{m}$ in $\bar{\bar{N}}$ such that$\bar{\varphi}(\bar{i}) \geq \bar{n}$ for every $i \geq \bar{m}$ in $\bar{\bar{N}}$.

Example 4.17

If $\bar{x} = \left(\frac{1}{n+1}, \frac{1}{n}, \frac{1}{n-1} \right)$ be fuzzy sequence in $\bar{R}$, then $\bar{y} = \left(\frac{1}{2n}, \frac{1}{2n-1}, \frac{1}{2n-2} \right)$ is fuzzy subsequence of $\bar{x}$. To show that, if we define $\bar{\varphi} : \bar{N} \to \bar{N}$ such that $\bar{\varphi} = \overline{2n} - 1$then

$(\bar{x} \circ \bar{\varphi})(\bar{n}) = \bar{x}(\bar{\varphi}(\bar{n})) = \bar{x}(\overline{2n-1}) = \left(\frac{1}{2n}, \frac{1}{2n-1}, \frac{1}{2n-2} \right)$will.

Definition 4.18.[1986 , ماكدوك]

The Field Axioms**:-**Let the triple $(F,+,\cdot)$consists of the non empty set F,with two binary operations , the addition $(+)$and the multiplication $(\cdot)$. The triple $(F,+,\cdot)$is said to bea field if satisfy the following:-

For all $x, y, z \in F$, $(x+y)+z = x+(y+z)$.

2)There is $0 \in F$ satisfy $x+0 = 0+x = x$, for all $x \in F$. 0 is said to be the additive identity in F.

3) $x+(-x) = (-x)+x = 0$. $(-x)$ is said to be the additivite inverse for x.

6)There is $1 \in F$ satisfy$1 \cdot x = x \cdot 1 = x$, for all $x \in F$ .1is said to be the multiplicative identity in F.

7) For all $x, y \in F$, then $x \cdot y = y \cdot x$, that is , the multiplication operation is commutative .

, there is $x \cdot x^{-1} = x^{-1} \cdot x = 1$. x^{-1} is said to be the multiplicative inverse for x.

9) For all $x, y, z \in F$, $(x+y) \cdot z = x \cdot z + y \cdot z$.

10) $1 \neq 0$.

Note that:-If we have a family of the fuzzy numbers .The triple $(\bar{F},+,\cdot)$ will be called the fuzzy field , if satisfy the following:-

1) For all $\bar{x}, \bar{y}, \bar{z} \in \bar{F}$, $(\bar{x}+\bar{y})+\bar{z} = \bar{x}+(\bar{y}+\bar{z})$

2) There is $\bar{0} \in \bar{F}$ satisfy $\bar{x}+\bar{0} = \bar{0}+\bar{x} = \bar{x}$, for all $\bar{x} \in \bar{F}$. $\bar{0}$ will be called the additive fuzzy identity in $\bar{F}$.

3) For all $\bar{x} \in \bar{F}$, there is $-\bar{x} \in \bar{F}$, such that $\bar{x}$ $\bar{x}$ $\bar{x}$ $\bar{x} = 0$ $\bar{x}$ fuzzy inverse for $\bar{x}$.

4) For all $\bar{x}, \bar{y} \in \bar{F}$, then $\bar{x}+\bar{y} = \bar{y}+\bar{x}$, that is , the additivite operation $(+)$ is commutative on $\bar{\bar{F}}$.

5) For all $\bar{x}, \bar{y}, \bar{z} \in \bar{F}$, $(\bar{x} \cdot \bar{y}) \cdot \bar{z} = \bar{x} \cdot (\bar{y} \cdot \bar{z})$

6) There is$\bar{1} \in \bar{F}$ satisfy$\bar{1} \cdot \bar{x} = \bar{x} \cdot \bar{1} = \bar{x}$, for all $\bar{x} \in \bar{F}$ .1 be called the multiplicative fuzzy identity in $\bar{\bar{F}}$.

7)For all $\bar{x}, \bar{y} \in \bar{F}$, then $\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$, that is , the multiplication operation $(\cdot)$is commutative on $\bar{\bar{F}}$.

8) For all $(\bar{x} \neq \bar{0}) \in \bar{\bar{F}}$, there is $\bar{x}^{-1}$ in , such that $\bar{x}^{-1}$ will be called

9) For all

10) $1 \neq \underline{0}$.

Definition 4.19.[Balmohan , 2004]

A linear space (or a vector space) over field K (R, C), is a non empty set X along w

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additivite operation $(+)$is commutative .

5) For all $x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

multiplication $\cdot : K \times X \to X$, such that, for all $x, y, z \in X$, and $a, b \in K$, we have

1) $x + y = y + x$

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2) $x+(y+z)=(x+y)+z$

3) There exists $0\in X$, such that, $x+0=x$.

α-cut of $Q(\bar{x}_2, \bar{y}_2)$ at degree α is

4) There exists $-x\in X$, such that, $x+(-x)=0$.

$d_\alpha(P_\alpha(x_{1,\alpha}, y_{1,\alpha}), Q_\alpha(x_{2,\alpha}, y_{2,\alpha})) = \sqrt{\Delta x_\alpha^2 + \Delta y_\alpha^2} = \sqrt{(x_{2,\alpha} - x_{1,\alpha})^2 + (y_{2,\alpha} - y_{1,\alpha})^2}$

5) $a\cdot(x+y)=a\cdot x+a.y$

6)

7) $(a\cdot b)\cdot x=a\cdot(b\cdot x)$

8) $1\cdot x=x$

Note that:-If we have a fuzzy field $\overline{K}(\bar{R}, \bar{C})$. Then the linear fuzzy space will be a family of fuzzy numbers $\overline{X}$ with addition operation $+: \overline{X}\times\overline{X}\to\overline{X}$, and fuzzy scalar multiplication $\cdot: \overline{\overline{K}}\times\overline{\overline{X}}\to\overline{\overline{X}}$, such that for all $\bar{x}, \bar{y}, \bar{z}\in\overline{\overline{X}}$, and $\bar{a}, \bar{b}\in\overline{\overline{K}}$, we have

1) $\bar{x}+\bar{y}=\bar{y}+\bar{x}$

2) $\bar{x}+(\bar{y}+\bar{z})=(\bar{x}+\bar{y})+\bar{z}$

3) There exists $\bar{0}\in\overline{\overline{X}}$, such that, .

4) There exists $-\bar{x}\in\overline{\overline{X}}$, such that, $\bar{x}+(-\bar{x})=\bar{0}$.

5)

6) $(\bar{a}+\bar{b})\cdot\bar{x}=\bar{a}\cdot\bar{x}+\bar{b}\cdot\bar{x}$

7) $(\bar{a}\cdot\bar{b})\cdot\bar{x}=\bar{a}\cdot(\bar{b}\cdot\bar{x})$

8) $\bar{1}\cdot\bar{x}=\bar{x}$

5. The Fuzzy Metric and The Fuzzy Pseudo-Metric

Definition 5.1.[Thomas , 2000]

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$d(P(x_1, y_2), Q(x_2, y_2)) = \sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Note that:-If we have the fuzzy plane $\overline{\overline{R}}^2$, and $\bar{P}(\bar{x}_1, \bar{y}_1), \bar{Q}(\bar{x}_2, \bar{y}_2)$ be two fuzzy ordered pairs in $\overline{\overline{R}}^2$. Let $x_{i,\alpha}\in\bar{x}_i[\alpha]$ of $\bar{x}_i$, and $y_{i,\alpha}\in\bar{y}_i[\alpha]$ of $\bar{y}_i$, $i=1,2$. The distance between two points $P_\alpha(x_{1,\alpha}, y_{1,\alpha})\in\bar{P}(\bar{x}_1[\alpha], \bar{y}_1[\alpha]) = \bar{P}(\bar{x}_1, \bar{y}_2)[\alpha]$ α-cut of and $Q_\alpha(x_{2,\alpha}$

and

$d_\alpha(P_\alpha(x_{1,\alpha}, y_{1,\alpha}), Q_\alpha(x_{2,\alpha}, y_{2,\alpha})) \in d(\bar{P}(\bar{x}_1, \bar{y}_1)[\alpha], \bar{Q}(\bar{x}_2, \bar{y}_2)[\alpha]) = d(\bar{P}(\bar{x}_1, \bar{y}_1), \bar{Q}(\bar{x}_2, \bar{y}_2))[\alpha]$

α-cut of $d(\bar{P}(\bar{x}_1, \bar{y}_1), \bar{Q}(\bar{x}_2, \bar{y}_2))$. Hence, the fuzzy distance between two fuzzy ordered pairs $\bar{P}(\bar{x}_1, \bar{y}_1)$ and $\bar{Q}(\bar{x}_2, \bar{y}_2)$ is

$$\bar{d}(\bar{P}(\bar{x}_1, \bar{y}_1), \bar{Q}(\bar{x}_2, \bar{y}_2)) = \sqrt{(\Delta \bar{x})^2 + (\Delta \bar{y})^2} = \sqrt{(\bar{x}_2 - \bar{x}_1)^2 + (\bar{y}_2 - \bar{y}_1)^2}$$

Definition 5.2. [Thomas , 2000]

The absolute value of a number x , denoted by $|x|$ is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Note that:-If we have the family of fuzzy real numbers $\bar{\bar{R}}$, and $\bar{x} \in \bar{\bar{R}}$ .Let $x_\alpha \in \bar{x}[\alpha]$ of $\bar{x}$, then the absolute value of x_α at degree α will be defined by

$$|x_\alpha|_\alpha = \begin{cases} x_\alpha, & x_\alpha \geq 0 \\ -x_\alpha, & x_\alpha < 0 \end{cases}$$

And $|x_\alpha|_\alpha \in |\bar{x}|[\alpha]$ α-cut of $|\bar{x}|$. Hence, the absolute fuzzy value of $\bar{x}$ is

$$|\bar{x}| = \begin{cases} \bar{x}, & \bar{x} \geq \bar{0} \\ -\bar{x}, & \bar{x} < \bar{0} \end{cases}$$

Definition 5.3.[Royden , 1966]

Let X be any set , a function $d : X \times X \to R$ is said to be a metric on X if:-

1) $d(x, y) \geq 0$, for all $x, y \in X$.
2) $d(x, y) = 0$ iff $x = y$.

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3) $d(x,y)=d(y,x)$, for all $x,y\in X$.

4) $d(x,y)\le d(x,z)+d(z,y)$, for all $x,y,z\in X$.

A set X with a metric d is said to be a metric space, and may be denoted (X,d).

Note:- If we have a family of fuzzy numbers $\bar{\bar{X}}$. A fuzzy function will be called a fuzzy metric on X if satisfy:-

1) $\bar{d}(\bar{x},\bar{y})\ge 0$, for all $\bar{x},\bar{y}\in\bar{X}$.

2) $\bar{d}(\bar{x},\bar{y})=0$ iff $\bar{x}=\bar{y}$.

all $x,y\in R$.

3) $\bar{d}(\bar{x},\bar{y})=\bar{d}(\bar{y},\bar{x})$, for all $\bar{x},\bar{y}\in\bar{X}$.

4) $\bar{d}(\bar{x},\bar{y})\le\bar{d}(\bar{x},\bar{z})+\bar{d}(\bar{z},\bar{y})$, for all $\bar{x},\bar{y},\bar{z}\in\bar{X}$.

A family of fuzzy numbers X with a fuzzy metric $\bar{d}$ will be called a fuzzy metric space , and will be denoted $.d$

Definition 5.4. [Sharma , 1977]

A function $d: X\times X\to R$ is called a pseudo-metric (semi-metric)for X iff it satisfies the axioms (1),(3),(4) of the first part of definition 5.3 , and the axiom $d(x,x)=0$, for all $x\in X$.

Note that:-A fuzzy mapping $\bar{\bar{d}}:\bar{\bar{X}}\times\bar{\bar{X}}\to\bar{\bar{R}}$ is said to be fuzzy pseudo-metric (fuzzy semi-metric) for $\bar{\bar{X}}$ iff it satisfies the axioms (1),(3),(4) of the second part of definition 5.3, and the axiom $\bar{d}(\bar{x},\bar{x})=0$, for all $\bar{x}\in\bar{X}$.

Remark 5.5

For a fuzzy pseudo-metric , we may have $\bar{d}(\bar{x},\bar{y})=0$ even if $\bar{x}\ne\bar{y}$. Thus for a fuzzy pseudo-metric $\bar{x}=\bar{y}\Rightarrow\bar{d}(\bar{x},\bar{y})=0$ but not conversely . It follows that every fuzzy metric is a fuzzy pseudo-metric but a fuzzy pseudo-metric is not necessarily fuzzy metric .

2) $d_\alpha(x_\alpha,y_\alpha)=|x_\alpha-y_\alpha|=0$ iff $x_\alpha=y_\alpha$ for all $x_\alpha\in\bar{x}[\alpha]$ of $\bar{x}$, ... Thus $\bar{d}(\bar{x},\bar{y})$

3) $d_\alpha(x_\alpha,y_\alpha)=|x_\alpha-y_\alpha|=|y_\alpha-x_\alpha|=d_\alpha(y_\alpha,x_\alpha)$, for all $x_\alpha\in\bar{x}[\alpha]$ of $\bar{x}$, $y_\alpha\in\bar{y}[\alpha]$

4) $d_\alpha(x_\alpha,y_\alpha)=|x_\alpha-z_\alpha+z_\alpha-y_\alpha|\le|x_\alpha-z_\alpha|+|z_\alpha-y_\alpha|=d_\alpha(x_\alpha,z_\alpha)+d_\alpha(z_\alpha,y_\alpha)$, for all $x_\alpha\in\bar{x}[\alpha]$ of $\bar{x}$, $y_\alpha\in\bar{y}[\alpha]$ of $\bar{y}$, $z_\alpha\in\bar{z}[\alpha]$ of $\bar{z}$ .Thus

$\bar{d}(\bar{x},\bar{y})=|\bar{x}-\bar{y}|\le|\bar{x}-\bar{z}|+|\bar{z}-\bar{y}|=\bar{d}(\bar{x},\bar{z})+\bar{d}(\bar{z},\bar{y})$ for all $\bar{x},\bar{y},\bar{z}\in\bar{R}$. Hence , $(\bar{R},\bar{d})$ is fuzzy metric space .

Remark 5.7

Above fuzzy metric will be called the fuzzy usual metric .

Example 5.8

Let the fuzzy mapping $\bar{d}:\bar{X}\times\bar{X}\to\bar{R}$ defined by

$$\bar{d}(\bar{x},\bar{y})=\begin{cases}0 , & \bar{x}=\bar{y} ,\\ 1 , & \bar{x}\ne\bar{y}\end{cases}$$

then

1) $d_\alpha(x_\alpha,y_\alpha)\ge 0$, for all $x_\alpha\in\bar{x}[\alpha]$ of $\bar{x}$, $y_\alpha\in\bar{y}[\alpha]$ of $\bar{y}$ .Thus $\bar{d}(\bar{x},\bar{y})\ge 0$, for all $\bar{x},\bar{y}\in\bar{X}$.

2) $d_\alpha(x_\alpha,y_\alpha)=0$ iff , for all $x_\alpha\in\bar{x}[\alpha]$ o

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y .Thus

x, y_α $\in y$ d x y 0 iff $x = y$.

Let the fuzzy mapping $\bar{d} : \bar{R} \times \bar{R} \to \bar{R}$ be 3) $d_\alpha(x_\alpha, y_\alpha) = 0 = d_\alpha(y_\alpha, x_\alpha)$, as $x_\alpha = y_\alpha$,

defined by $\bar{d}(\bar{x}, \bar{y}) = \overline{|x - y|}$, for all $\bar{x}, \bar{y} \in \bar{R}$ .To show and $d_\alpha(x_\alpha, y_\alpha) = 1 = d_\alpha(y_\alpha, x_\alpha)$,as $x_\alpha \neq y_\alpha$

for all x $\in$

1) $d_\alpha(x_\alpha, y_\alpha) = |x_\alpha - y_\alpha|$

$y_\alpha \in \bar{y}[\alpha]$ of $\bar{y}$. Thus $\bar{d}(\bar{x}, \bar{y}) = 1 = \bar{d}(\bar{y}, \bar{x})$,

$\bar{d}(\bar{x}, \bar{y}) = \overline{|x - y|} \geq \bar{0}$, for all $\bar{x}, \bar{y} \in \bar{R}$.

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4) If $x_\alpha = y_\alpha$, then $d_\alpha(x_\alpha, y_\alpha) = 0$.

Also $d_\alpha(x_\alpha, z_\alpha) \geq 0$ and $d_\alpha(z_\alpha, y_\alpha) \geq 0$. Thus $d_\alpha(x_\alpha, y_\alpha) \leq d_\alpha(x_\alpha, z_\alpha) + d_\alpha(z_\alpha, y_\alpha)$. If $x_\alpha \neq y_\alpha$, then $d_\alpha(x_\alpha, y_\alpha)$ is different from at least one of and y_α, so at least one of $d_\alpha(x_\alpha, z_\alpha)$ and $d_\alpha(z_\alpha, y_\alpha)$ must be equal to 1.

Thus $d_\alpha(x_\alpha, y_\alpha) \leq d_\alpha(x_\alpha, z_\alpha) + d_\alpha(z_\alpha, y_\alpha)$, for all $x_\alpha \in \bar{x}[\alpha]$ of $\bar{x}$, $y_\alpha \in \bar{y}[\alpha]$ of $\bar{y}$, $z_\alpha \in \bar{z}[\alpha]$ of $\bar{z}$.

Therefore $\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$, for all $\bar{x}, \bar{y}, \bar{z} \in \bar{R}$.

Hence $(\bar{\bar{x}}, \bar{\bar{d}})$ is fuzzy metric space.

Remark 5.9

discrete metric.

Example 5.10

Let X be the family of all fuzzy functions into itself. Let the fuzzy mapping $d : \bar{X} \times \bar{X} \to \bar{R}^0$ be defined by

$$\bar{d}(\bar{f}, \bar{g}) = \sup |\bar{f}(\bar{x}) - \bar{g}(\bar{x})|,$$

To show that d is fuzzy metric for X, then

1) $d_\alpha(f_\alpha, g_\alpha) = \sup |f_\alpha(x_\alpha) - g_\alpha(x_\alpha)|_\alpha \geq 0$, for all $f_\alpha(x_\alpha) \in \bar{f}(\bar{x})[\alpha]$ of $\bar{f}(\bar{x})$, $g_\alpha(x_\alpha) \in \bar{g}(\bar{x})[\alpha]$ of $\bar{g}(\bar{x})$, $x_\alpha \in \bar{x}[\alpha]$ of $x, y \in S$ $\bar{x}$. Thus $\bar{d}(\bar{f}, \bar{g}) = \sup |\bar{f}(\bar{x}) - \bar{g}(\bar{x})| \geq \bar{0}$, for all $\bar{f}, \bar{g} \in \bar{X}$.

2) $d_\alpha(f_\alpha, g_\alpha) = \sup |f_\alpha(x_\alpha) - g_\alpha(x_\alpha)|_\alpha = 0$ iff $f_\alpha(x_\alpha) = g_\alpha(x_\alpha)$, for all $f_\alpha(x_\alpha) \in \bar{f}(\bar{x})[\alpha]$ of $\bar{f}(\bar{x})$, $g_\alpha(x_\alpha) \in \bar{g}(\bar{x})[\alpha]$ of $\bar{g}(\bar{x})$, $x_\alpha \in \bar{x}[\alpha]$ of $\bar{x}$. Thus $\bar{d}(\bar{f}, \bar{g}) = \sup |\bar{f}(\bar{x}) - \bar{g}(\bar{x})| = \bar{0}$ iff $\bar{f} = \bar{g}$.

3) $d_\alpha(f_\alpha, g_\alpha) = \sup |f_\alpha(x_\alpha) - g_\alpha(x_\alpha)|_\alpha = \sup |g_\alpha(x_\alpha) - f_\alpha(x_\alpha)|_\alpha = d_\alpha(g_\alpha, f_\alpha)$

$$\bar{d}(\bar{f}, \bar{g}) = \sup |\bar{f}(\bar{x}) - \bar{g}(\bar{x})| = \sup |\bar{g}(\bar{x}) - \bar{f}(\bar{x})| = \bar{d}(\bar{g}, \bar{f}),$$

for all $\bar{f}, \bar{g} \in \bar{X}$.

4) $d_\alpha(f_\alpha, g_\alpha) = \sup |f_\alpha(x_\alpha) - g_\alpha(x_\alpha)|_\alpha = \sup |f_\alpha(x_\alpha) - h_\alpha(x_\alpha) + h_\alpha(x_\alpha) - g_\alpha(x_\alpha)|_\alpha$
$\leq \sup(|f_\alpha(x_\alpha) - h_\alpha(x_\alpha)| + |h_\alpha(x_\alpha) - g_\alpha(x_\alpha)|)$.
$\leq \sup |f_\alpha(x_\alpha) - h_\alpha(x_\alpha)|_\alpha + \sup |h_\alpha(x_\alpha) - g_\alpha(x_\alpha)|_\alpha = d_\alpha(f_\alpha, h_\alpha) + d_\alpha(h_\alpha, g_\alpha)$

for all $f_\alpha(x_\alpha) \in \bar{f}(\bar{x})[\alpha]$ of $\bar{f}(\bar{x})$, $g_\alpha(x_\alpha) \in \bar{g}(\bar{x})[\alpha]$ of $\bar{g}(\bar{x})$, $h_\alpha(x_\alpha) \in \bar{h}(\bar{x})[\alpha]$ of $\bar{h}(\bar{x})$, $x_\alpha \in \bar{x}[\alpha]$ of $\bar{x}$. Thus

$$\bar{d}(\bar{f}, \bar{g}) = \sup |\bar{f}(\bar{x}) - \bar{g}(\bar{x})| \leq \sup |\bar{f}(\bar{x}) \ldots = \bar{d}(\bar{f}, \bar{h}) + \bar{d}(\ldots$$

for all $\bar{f}, \bar{g}, \bar{h} \in \bar{X}$. Hence $(\bar{X}, \bar{d})$ is fuz... space.

Example 5.11

... fuzzy sequence $\bar{x} = \bar{x}_n$. Defin...

$$\bar{d}(\bar{x}, \bar{y}) = \sum_{n=1}^{\infty} \frac{|\bar{x}_n - \bar{y}_n|}{2^n\left[1 + |\bar{x}_n - \bar{y}_n|\right]},$$

$x_{i,\alpha} = x_{in,\alpha} \in \bar{x}[\alpha] = \bar{x}_n[\alpha]$ of $\bar{x} = \bar{x}_n$, $y_{i,\alpha} = y_{in,\alpha} \in \bar{y}[\alpha] = \bar{y}_n[\alpha]$ of $\bar{y} = \bar{y}_n$, $z_{i,\alpha} = y_{in,\alpha} \in \bar{z}[\alpha] = \bar{z}_n[\alpha]$ of $\bar{z} = \bar{z}_n$, then

$\langle \quad \rangle$

, for all 1)

$f_\alpha(x_\alpha) \in \bar{f}(\bar{x})[\alpha]$ of

$\bar{f}\ \bar{x}$, $g\ (x\ ) \in \bar{g}\ \bar{x}\ [\alpha]$ of $\bar{g}(\bar{x})$, $x \in \bar{x}\ [\alpha]$ of $\bar{x}$ .Thus

Since $\sum_{n=1}^{\infty} \frac{|x_{in,\alpha} - y_{in,\alpha}|}{2^n[1+|x_{in,\alpha} - y_{in,\alpha}|_\alpha]} \in$

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$$\left(\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n \left[1 + |x_n - y_n|\right]} \right)_{[\alpha]}$$ α -cut of

$$\sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n \left[1 + |x_n - y_n|\right]}$$. Then , for all $x, y \in S$

we have $d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n \left[1 + |x_n - y_n|\right]} \geq 0$.

2)

for all $\bar{x}, \bar{y} \in \bar{\bar{S}}$.

4) $$d_{i,\alpha}(x_{i,\alpha}, y_{i,\alpha}) = \sum_{n=1}^{\infty} \frac{|x_{in,\alpha} - y_{in,\alpha}|_{\alpha}}{2^n[1+|x_{in,\alpha} - y_{in,\alpha}|_{\alpha}]}$$

$$= \sum_{n=1}^{\infty} \frac{|x_{in,\alpha} - z_{in,\alpha} + z_{in,\alpha} - y_{in,\alpha}|_{\alpha}}{2^n[1+|x_{in,\alpha} - z_{in,\alpha} + z_{in,\alpha} - y_{in,\alpha}|_{\alpha}]}$$

$$\leq \sum_{n=1}^{\infty} \frac{|x_{in,\alpha} - z_{in,\alpha}|_{\alpha}}{2^n[1+|x_{in,\alpha} - z_{in,\alpha}|_{\alpha}]} + \sum_{n=1}^{\infty} \frac{|z_{in,\alpha} - y_{in,\alpha}|_{\alpha}}{2^n[1+|z_{in,\alpha} - y_{in,\alpha}|_{\alpha}]}$$

$$= d_{i,\alpha}(x_{i,\alpha}, z_{i,\alpha}) + d_{i,\alpha}(z_{i,\alpha}, y_{i,\alpha})$$

$$d_{i,\alpha}(x_{i,\alpha}, y_{i,\alpha}) = \sum_{n=1}^{\infty} \frac{|x_{in,\alpha} - y_{in,\alpha}|_{\alpha}}{2^n[1+|x_{in,\alpha} - y_{in,\alpha}|_{\alpha}]} = 0 \Leftrightarrow x_{in,\alpha} = y_{in,\alpha}$$ for all $i, n \in N$.

Since $d_{i,\alpha}(x_{i,\alpha}, y_{i,\alpha}) \in \bar{d}(\bar{\bar{x}}, \bar{\bar{y}})[\alpha]$ of $\bar{d}(\bar{\bar{x}}, \bar{\bar{y}})$. Then

$$\bar{d}(\bar{\bar{x}}, \bar{\bar{y}}) = \sum_{n=1}^{\infty} \frac{|\bar{x}_n - \bar{y}_n|}{\bar{2}^n\left[\bar{1} + |\bar{x}_n - \bar{y}_n|\right]} = \bar{0} \Leftrightarrow \bar{x}_n = \bar{y}_n, \ n \in N$$

$\Leftrightarrow \bar{\bar{x}} = \bar{\bar{y}}$

Since $\bar{d}(\bar{x},\bar{y})$, $d_{i,\alpha}(x_{i,\alpha}, z_{i,\alpha}) \in \bar{d}(\bar{x},\bar{z})[\alpha]$ of $\bar{d}(\bar{x},\bar{z})$, and $d_{i,\alpha}(z_{i,\alpha}, y_{i,\alpha}) \in \bar{d}(\bar{z},\bar{y})[\alpha]$ of $\bar{d}(\bar{z},\bar{y})$. Then

3) $$d_{i,\alpha}(x_{i,\alpha}, y_{i,\alpha}) = \sum_{n=1}^{\infty} \frac{|x_{in,\alpha} - y_{in,\alpha}|_{\alpha}}{2^n[1+|x_{in,\alpha} - y_{in,\alpha}|_{\alpha}]} = \sum_{n=1}^{\infty} \frac{|y_{in,\alpha} - x_{in,\alpha}|_{\alpha}}{2^n[1+|y_{in,\alpha} - x_{in,\alpha}|_{\alpha}]} = d_{i,\alpha}(y_{i,\alpha}, x_{i,\alpha})$$

.

Since $d_{i,\alpha}(x_{i,\alpha}, y_{i,\alpha}) \in \bar{d}(\bar{\bar{x}},\bar{\bar{y}})[\alpha]$ of $\bar{d}(\bar{\bar{x}},\bar{\bar{y}})$, and $d_{i,\alpha}(y_{i,\alpha}, x_{i,\alpha}) \in d(\bar{y},\bar{x})[\alpha]$ of $d(\bar{y},\bar{x})$. Then

$$\bar{d}(\bar{\bar{x}},\bar{\bar{y}}) = \sum_{n=1}^{\infty} \frac{\left|\bar{\bar{x}}_n - \bar{\bar{y}}_n\right|}{2^n\left[\bar{1} + \left|\bar{\bar{x}}_n - \bar{\bar{y}}_n\right|\right]} = \sum_{n=1}^{\infty} \frac{\left|\bar{\bar{y}}_n - \bar{\bar{x}}_n\right|}{2^n\left[\bar{1} + \left|\bar{\bar{y}}_n - \bar{\bar{x}}_n\right|\right]} = \bar{d}(\bar{\bar{y}},\bar{\bar{x}})$$

for all $\bar{x}, \bar{y}, \bar{z} \in \bar{\bar{S}}$. Hence $\bar{\bar{d}}$ is a fuzzy metric on $\bar{\bar{S}}$, that is, $(\bar{\bar{S}}, \bar{\bar{d}})$ is a fuzzy metric space.

Example 5.12

Let $\left\{\left(X_i, \bar{\bar{d_i}}\right) : i = 1,\ldots,n\right\}$ be a finite class of fuzzy metric spaces .To show that each of the fuzzy functions $\bar{d}$ and $\bar{\rho}$ defined as follows are fuzzy metrics on the fuzzy product $\bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$

$i)$

$$\bar{d}\left(\left(\bar{x}_1,\ldots,\bar{x}_n\right),\left(\bar{y}_1,\ldots,\bar{y}_n\right)\right) = \max \bar{d}_i\left(\bar{x}_i,\bar{y}_i\right),\ i=1,\ldots,n$$

$ii)$

$$\bar{d}\left(\left(\bar{x}_1,\ldots,\bar{x}_n\right),\left(\bar{y}_1,\ldots,\bar{y}_n\right)\right) = \sum_{i=1}^{n} \bar{d}_i\left(\bar{x}_i,\bar{y}_i\right)$$

Then

$i)$ Let $\bar{x} = \left(\bar{x}_1,\ldots,\bar{x}_n\right), \bar{y} = \left(\bar{y}_1,\ldots,\bar{y}_n\right), \bar{z} = \left(\bar{z}_1,\ldots,\bar{z}_n\right) \in \bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$, then

For each $\bar{\bar{d}}_i, i=1,\ldots,n$, we have $\bar{d}_i\left(\bar{x}_i,\bar{y}_i\right) \geq \bar{0}$, then $\max \bar{d}_i\left(\bar{x}_i,\bar{y}_i\right) \geq \bar{0}$. Thus

$\bar{d}\left(\bar{x},\bar{y}\right) \geq \bar{0}$, for all .

2) For each $\bar{\bar{d}}_i, i=1,\ldots,n$, we have $\bar{d}_i\left(\bar{x}_i,\bar{y}_i\right) = \bar{0}$ iff $\bar{x}_i = \bar{y}_i$, $i=1,\ldots,n$. It follows that

$$\max \bar{d}_i\left(\bar{x}_i,\bar{y}_i\right) = \bar{0} \Leftrightarrow \bar{x}_i = \bar{y}_i,\ i=1,\ldots,n$$

$$\Leftrightarrow \left(\bar{x}_1,\ldots,\bar{x}_2\right) = \left(\bar{y}_1,\ldots,\bar{y}_n\right)$$

$$\Leftrightarrow \bar{x} = \bar{y}$$

Thus $\bar{d}\left(\bar{x},\bar{y}\right) = \bar{0}$ iff $\bar{x} = \bar{y}$.

for all $\bar{x}, \bar{z}, \bar{y} \in \bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$. Hence , $\bar{\bar{d}}$ is a fuzzy metric for .

ii) Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n), \bar{y} = (\bar{y}_1, \ldots, \bar{y}_n), \bar{z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$, then

1) $\bar{\rho}(\bar{x}, \bar{y}) = \sum_{i=1}^{n} \bar{\rho}_i(\bar{x}_i, \bar{y}_i) \geq \bar{0}$, since $\bar{\rho}_i(\bar{x}_i, \bar{y}_i) \geq 0$ for each $i = 1, \ldots, n$., for all $\bar{x}, \bar{y} \in \bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$.

2) $\bar{\rho}(\bar{x}, \bar{y}) = \sum_{i=1}^{n} \bar{\rho}_i(\bar{x}_i, \bar{y}_i) = \bar{0} \Leftrightarrow \bar{\rho}_i(\bar{x}_i, \bar{y}_i) = 0$, $i = 1, \ldots, n$

$\Leftrightarrow \quad \bar{x}_i = \bar{y}_i$, $i = 1, \ldots, n$

$\Leftrightarrow \quad (\bar{x}_1, \ldots, \bar{x}_n) = (\bar{y}_1, \ldots, \bar{y}_n)$

$\Leftrightarrow \quad \bar{x} = \bar{y}$

3) $\bar{\rho}(\bar{x}, \bar{y}) = \sum_{i=1}^{n} \bar{\rho}_i(\bar{x}_i, \bar{y}_i) = \sum_{i=1}^{n} \bar{\rho}_i(\bar{y}_i, \bar{x}_i) = \bar{\rho}(\bar{y}, \bar{x})$, for all $\bar{x}, \bar{y} \in \bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$.

4) For each $\bar{\bar{\rho}}_i, i = 1, \ldots, n$, we have $\bar{\rho}_i(\bar{x}_i, \bar{y}_i) \leq \bar{\rho}_i(\bar{x}_i, \bar{z}_i) + \bar{\rho}_i(\bar{z}_i, \bar{y}_i)$.

Thus

$$\bar{\rho}(\bar{x}, \bar{y}) = \sum_{i=1}^{n} \bar{\rho}_i(\bar{x}_i, \bar{y}_i) \leq \sum_{i=1}^{n} \bar{\rho}_i(\bar{x}_i, \bar{z}_i) + \sum_{i=1}^{n} \bar{\rho}_i(\bar{z}_i, \bar{y}_i) = \bar{\rho}(\bar{x}, \bar{z}) + \bar{\rho}(\bar{z}, \bar{y})$$

$\bar{d}(\bar{x}, \bar{y}) = \max \bar{d}_i(\bar{x}_i, \bar{y}_i) = \max \bar{d}_i(\bar{y}_i, \bar{x}_i) = \bar{d}(\bar{y}, \bar{x})$, for all $\bar{x}, \bar{y} \in \bar{\bar{X}}_1 \times \ldots \times \bar{\bar{X}}_n$.

4) For each $\bar{\bar{d}}_i, i = 1, \ldots, n$, we have $\bar{d}_i(\bar{x}_i, \bar{y}_i) \leq \bar{d}_i(\bar{x}_i, \bar{z}_i) + \bar{d}_i(\bar{z}_i, \bar{y}_i)$. Then

$\max \bar{d}_i(\bar{x}_i, \bar{y}_i) \leq \max \bar{d}_i(\bar{x}_i, \bar{z}_i) + \max \bar{d}_i(\bar{z}_i, \bar{y}_i)$. Thus $d(x, y) \leq d(x, z) + d(z, y)$,

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for all $x, y \in X_1 \times \ldots \times X_n$. Hence , ρ is fuzzy metric for $X_1 \times \ldots \times X_n$.

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