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An Interesting q-Continued Fractions of Ramanujan

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Abstract— **In this paper, we establish an interesting q-identity and an integral representation of a q-continued fraction of Ramanujan. We also compute explicit evaluation of this continued fraction and derive its relation with Ramanujan G¨ollnitz -Gordon continued fraction. 2000 Mathematics Subject Classification: 11A55.**
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1 Introduction

Ramanujan a pioneer in the theory of continued fraction has recorded several in the process rediscovered few continued fractions found earlier by Gauss, Eisenstein and Rogers in his notebook [10]. In fact Chapter 12 and Chapter 16 of his Second Notebook [10] is devoted to continued fractions. Proofs of these continued fractions over years are given by several mathematician, we mention here specially G.E. Andrews[3], C. Adiga, S. Bhargava and G.N. Watson [1] whose works have been compiled in [4] and [5].

The celebrated Roger Ramanujan continued fraction is defined by

$$R(q) := \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots, |q| < 1.$$

On page 365 of his lost notebook [11], Ramanujan recorded five modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$ $R(q^4)$ and $R(q^5)$.

The well known Ramanujan's cubic continued fraction defined by

$$J(q) := \frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\cdots, |q| < 1.$$

[1]Supported by UGC Grant No. F41-1392/2012/(SR) is recorded on page 366 of his lost notebook [11]. Several new modular equation relating $J(q)$ with $J(-q)$, $J(q^2)$ and $J(q^3)$ are established by H.H. Chan [8].

Similarly the Ramanujan G¨ollnitz-Gordon continued fraction K(q) defined by

$$K(q) := \frac{q^{1/2}}{1+q+}\frac{q^2}{1+q^3+}\frac{q^4}{1+q^5+}\frac{q^6}{1+q^7+}\cdots, |q| < 1,$$

satisfies several beautiful modular relations. One may see traces of modular equation related to K(q) on page 229 of Ramanujan's lost notebook [11]. Further works related to K(q) in recent years have been done by various authors including Chan and S.S Huang [9] and K.R. Vasuki and B.R. Srivatsa Kumar [12].

Motivated by these works in this paper we study the Ramanujan continued fraction

$$M(q) := \frac{q^{1/2}}{1-q+}\frac{q(1-q)}{1+q^2+(1-q)(1+q^4)+}\frac{q(1-q^3)^2}{(1-q)(1+q^6)+}\frac{q(1-q^5)^2}{\cdots}, \quad |q| < 1$$

$(q^4; q^4)^2$

$$= q^{1/2}\frac{(\ldots)_\infty}{(q^2;q^4)^2_\infty}. \qquad (1.1)$$

In Chapter 16 Entry 12 of [5], Ramanujan has recorded the following continued fraction

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} = \frac{1}{1-ab+}\,\frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)+}\,\frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)+}\cdots,\quad |ab|<1,\ |q|<1. \qquad (1.2)$$

In fact setting $a = q^{1/2}$ and $b = q^{1/2}$ in (1.2), we obtain (1.1).

In Section 2 we obtain an interesting q-identity related to M(q) using Ramanujan's ${}_1\psi_1$ summation formula [5, Ch. 16, Entry 17]

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n = \frac{(az)_\infty(q/az)_\infty(q)_\infty(b/a)_\infty}{(z)_\infty(b/az)_\infty(b)_\infty(q/a)_\infty},\quad |b/a|<|z|<1, \qquad (1.3)$$

and Andrew's identity [4, p. 57],

$$\sum_{n=0}^{\infty}\frac{q^{kn}}{1-q^{ln+k}} = \sum_{n=0}^{\infty}q^{ln^2+2kn}\,\frac{1+q^{ln+k}}{1-q^{ln+k}}. \qquad (1.4)$$

In Section 3 we obtain several relation of M(q) with theta function $\phi(q)$, $\psi(q)$ and $\chi(q)$. In Section 4 we obtain an integral representation of M(q). In Section 5 we derive a formula that help us to obtain relation among $M(q^{1/2})$, M(q), $M(q^2)$ and $M(q^4)$. We establish explicit formulas for the evaluation of $M(e^{-\pi\sqrt{n}})$ and $M(e^{-\pi\sqrt{n/2}})$ in Section 6.

We conclude this introduction with few customary definition we make use in the sequel. For a and q complex number with $|q| < 1$

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

and

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a)_\infty}{(aq^n)_\infty},\quad n: \text{any integer}.$$

$$f(a,b) := \sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2} \qquad (1.5)$$

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$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1. \tag{1.6}$$

Identity (1.6) is the Jacobi's triple product identity in Ramanujan's notation [5, Ch. 16, Entry 19]. It follows from (1.5) and (1.6) that [5, Ch. 16, Entry 22],

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.8}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{1.9}$$

2 q-Identity related to M(q)

Theorem 2.1

$$M(q) = \sum_{n=0}^{\infty} q^{n(8n+4)+1/2}\frac{1+q^{8n+2}}{1-q^{8n+2}} - \sum_{n=0}^{\infty} q^{(n+1)(8n+4)+1/2}\frac{1+q^{8n+6}}{1-q^{8n+6}} \tag{2.1}$$

Proof: Changing q to q^8, then setting $a = q^2$, $b = q^{10}$ and $z = q^2$ in ${}_1\psi_1$ summation formula (1.3) we obtain

$$\frac{(q^4;q^4)^2_\infty}{(q^2;q^4)^2_\infty} = \sum_{n=0}^{\infty}\frac{q^{2n}}{1-q^{8n+2}} - \sum_{n=0}^{\infty}\frac{q^{6n+4}}{1-q^{8n+6}} \tag{2.2}$$

employing Andrews identity (1.4) with $k = 2$, $l = 8$ and $k = 6$, $l = 8$ in both the summations in right side of the identity (2.2) respectively and finally multiplying both sides of the resulting identity with $q^{1/2}$ and using product represtation of M(q) (1.1), we complete the proof of Theorem 2.1.

3 Some Identities involving M(q)

We obtain relation of M(q) in terms of theta function $\phi(q)$, $\psi(q)$ and $\chi(q)$.

Theorem 3.1

$M(q) = q^{1/2}\,\psi^{4}(q)\,,\ \phi^2(q$

$$8M(q^2) = \phi^2(q) - \phi^2(-q)\,,$$

$$16M^2(q) = \phi^4(q) - \phi^4(-q)\,,$$

$$\frac{M^2(q)}{M(q^2)} = \phi^2(q^2)\,,$$

$$4M(q^2) = \phi^2(q) - \phi^2(q^2)\,,$$

$$\frac{M^{-1}(q) + M(q)}{M^{-1}(q) - M(q)} = \frac{1 + q\psi^4(q^2)}{1 - q\psi^4(q^2)}\,,$$

$$\frac{\chi^2(q)}{}$$

$$8M(q^2) = \chi^2(-q) \qquad \varphi^2(-q^2) - \varphi^2(-q)\,.$$

Proof: Using [5, Ch. 16, Entry 22(ii)] in (1.1), we obtain

$$M(q) = q^{1/2}\psi^2(q^2)\,.$$

Employing [5, Ch. 16, Entry 25(iv)] in (3.8), we obtain (3.1). From (3.1) we have(3.1)

$$M(q^2) = q\,\frac{\psi^4(q^2)}{\phi^2(q^2)}\,. \tag{3.9}$$

Employing [5, Ch. 16, Entry 25(vii)] and [5, Ch. 16, Entry 25(vi)] in (3.9), we obtain (3.2). Identity (3.3) immediately follows from (3.8) and [5, Ch. 16, Entry 25(vii)]. Again from (3.1), we have

$$\frac{M^2(q)}{M(q^2)} = \frac{\psi^8(q)\phi^2(q^2)}{\psi^4(q^2)\phi^4(q)}\,, \tag{3.10}$$

employing [5, Ch. 16, Entry 25(iv)] in the identity (3.10) we obtain (3.4). From (3.2) and (3.3), we have

$$64M^2(q^2) + 16M^2(q) = 16\phi^2(q)M(q^2)\,, \tag{3.11}$$

dividing the identity (3.11) throughout by $16M(q^2)$ and using (3.4) we obtain (3.5). From (3.1) we deduce that

$$M^{-1}(q) + M(q) = \frac{\phi^4(q) + q\,\psi^8(q)}{q^{1/2}\phi^2(q)\psi^4(q)}\,, \tag{3.12}$$

and

$$M^{-1}(q) - M(q) = \frac{\phi^4(q) - q\,\psi^8(q)}{q^{1/2}\phi^2(q)\psi^4(q)}\,. \tag{3.13}$$

On dividing (3.12) by (3.13) and using [5, Ch. 16, Entry 25(iv)] in the resulting identity, we complete the proof of (3.6).

From (1.7) and (1.9) we have

$$\phi(-q) + \frac{\chi(q)}{\chi(-q)} \quad \frac{2}{\phi(-q\ \ )} = \frac{(q;q)_\infty}{(-q;q)_\infty} \quad 1 + \frac{f(q,q)}{f(-q,-q)}\,,$$

employing [5, Ch. 16, Entry 30(ii)] in right hand side of above identity we obtain

$$\phi(-q) + \frac{\chi(q)}{\chi(-q)}\phi(-q^2) = \frac{2(q^8;q^8)_\infty^5 (q^{32};q^{32})_\infty^2}{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^2 (q^{64};q^{64})_\infty^4} \frac{M(q^{16})}{q^8}. \qquad (3.14)$$

Again from (1.7) and (1.9) we have

$$\phi(-q) - \frac{\chi(q)}{\chi(-q)}\phi(-q^2) = \frac{(q;q)_\infty}{(-q;q)_\infty}\frac{f(q,q)}{1-f(-q,-q)},$$

employing [5, Ch. 16, Entry 30(ii)] and [5, Ch. 16, Entry 18(ii)] in right hand side of above identity we obtain

$$\phi(-q) - \frac{\chi(q)}{\chi(-q)}\phi(-q^2) = \frac{-4q(-q^8;q^8)_\infty(q^{64};q^{64})_\infty^3}{(-q^{16};q^{32})_\infty}\frac{q^8}{M(q^{16})}. \qquad (3.15)(3.16)(3.17)$$

Multiplying (3.14) and (3.15) we complete the proof of (3.7).

Theorem 3.2. Let $u = M(q)$, $v = M(-q)$ and $w = M(q^2)$, then

$$u^2 - v^2 = 8w^2$$

Proof: On substituting (3.4) in (3.5), we obtain

$$\phi^2(q) = 4M^2(q^2) + M^2(q) \,.\, M(q^2)$$

Changing q to $-q$ in (3.16), we have

$$\phi^2(-q) = 4M^2(q^2) + M^2(-q) \,.\, M(q^2)$$

Subtracting (3.17) from (3.16) and using identity (3.2), we complete the proof of Theorem 3.2.

4 Integral Representation of M(q)

Theorem 4.1. For $0 < |q| < 1$

$$M(q) = \exp \int_0^q \left(\frac{1}{2q} + \frac{4\,\phi^4(-q) - 1}{8q} + \frac{q\,\phi'(q)}{2\,\phi(q)} \right) dq \,, \qquad (4.1)$$

where $\phi(q)$ and $\psi(q)$ are as defined in (1.7) and (1.8). Proof: Taking log on both sides of (3.1), we have

$$\log M(q) = \tfrac{1}{2} \log q + 4 \log \psi(q) - 2 \log \phi(q) \,. \qquad (4.2)$$

Employing [5, Ch. 16, Entry 23(ii)] and [5, Ch. 16, Entry 23(i)] on right hand side of (4.2), we obtain

$$\log M(q) = \frac{1}{2} \log q + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{\quad} \,. \qquad (4.3)$$

Differentiating (4.3) and simplifying, we have

$$\frac{d}{dq} \log M(q) = \frac{1}{2q} + \frac{4}{q} \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(1+q^n)^2} + \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} \,. \qquad (4.4)$$

Using Jacobi's identity [5, Ch. 16, Identity 33.5, p. 54)] and [5, Ch. 16, Entry 23(i)] and integrating both sides and finally exponentiating both sides of identity (4.4), we complete the proof of Theorem 4.1.

5 Modular Equation of Degree n and Relation Between M(q) and M(q^n)

In the terminology of hypergeometric function, a modular equation of degree n is a relation between α and β that is induced by

$$n \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha)}{{}_2F_1(1/2, 1/2; 1; \alpha)} = \frac{{}_2F_1(1/2, 1/2; 1; 1-\beta)}{{}_2F_1(1/2, 1/2; 1; \beta)} \,,$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k \,,$$

and

$$(a)_k = \frac{(a+k)}{(a)} \,.$$

Let $Z_1(r) = {}_2F_1(1/r, r - 1/r; 1; \alpha)$ and $Z_n(r) = {}_2F_1(1/r, r - 1/r; 1; \beta)$, where n is the degree of the modular equation. The multiplier m(r) is defined by the equation

$$m(r) = \frac{Z_1(r)}{Z_n(r)} .$$

Theorem 5.1. If

$$q = \exp\left(-\pi \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha)}{{}_2F_1(1/2, 1/2; 1; \alpha)}\right),$$

then

$$\alpha = 16 \frac{M^4(q)}{M^4(q^{1/2})}$$

Proof: From (1.1) and (1.7), we have

$$M(q)\phi^2(q) = q^{1/2} \frac{(q^4; q^4)_\infty^2 \quad (-q; q^2)_\infty^4 (q^4; q^4)_\infty^2}{(q^2; q^4)_\infty^2 (-q^2; q^2)_\infty^4 (q^2; q^4)_\infty^2}$$
$$= M^2(q^{1/2}) .$$

Substituting (5.3) in (3.3), we obtain

$$16M^2(q) = \frac{M^4(q^{1/2})}{M^2(q)} \left(1 - \frac{\phi^4(-q)}{\phi^4(q)}\right) .$$

From a known identity [5, Ch. 16, p. 100, Entry 5] and (5.1) it is implied that

$$\alpha = 1 - \frac{\phi^4(-q)}{\phi^4(q)} . \qquad (5.5)$$

Using (5.5) in (5.4), we complete the proof of (5.2).

Let α and β be related by (5.1). If β has degree n over α then from Theorem 5.1, we obtain

$$\beta = 16 \frac{M^4(q^n)}{M^4(q^{n/2})} . \qquad (5.6)$$

Corollary 5.2. Let $u = M(q^{1/2})$, $v = M(q)$, $w = M(q^2)$ and $x = M(q^4)$, then

$$16x^4v^2 + 32x^3wv^2 - 4x^3wu^4 + 24x^2w^2v^2 + 8xw^3v^2 - xw^3u^4 + w^4v^2 = 0 . \qquad (5.7)$$

Proof: From [5, Entry 24(v), p. 216], we have

$$\sqrt{1-\alpha} = \left(\frac{1-\beta^{1/4}}{}\right)^2 . \qquad (5.8)$$

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On using (5.6) with n = 4 and (5.2) in (5.8), we obtain

$$\sqrt{\frac{u^4-16v^4}{u^4}} = \left(\frac{w-2x}{w+2x}\right)^2. \qquad (5.9)$$

Squaring both side of (5.9) and then simplifying, we obtain (5.7).

6 Evaluations of M(q)

As an application of Theorem 5.1, we establish few explicit evaluation of M(q).

Let $q_n = e^{-\pi\sqrt{n}}$ and let α_n denote the corresponding value of α in (5.1). Then by Theorem 5.1, we have

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n/2}})} = \left(\frac{1}{2\alpha_n}\right)^{1/4} \qquad (6.1)$$

From [5, Ch. 17, p. 97], we have $\alpha_1 = \frac{1}{2}$, $\alpha_2 = (\sqrt{2}-1)^2$ and $\alpha_4 = (\sqrt{2}-1)^4$.
Thus from (6.1), it immediately follows

$$\frac{M(e^{-\pi})}{M(e^{-\pi/\sqrt{2}})} = \left(\frac{1}{2}\right)^{5/4}, \qquad (6.2)$$

$$\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2}\sqrt{\sqrt{2}-1}, \qquad (6.3)$$

$$\frac{M(e^{-2\pi})}{M(e^{-\pi})} = \frac{\sqrt{2}-1}{2}. \qquad (6.4)$$

Ramanujan has recorded several modular equation in his notebook [10, p. 204-237] and [10, p. 156-160] which are very useful in the computation of class invariants and the values of theta function. Ramanujan has also recorded several values of theta function $\phi(q)$ and $\psi(q)$ in his notebook. For example

$\phi(e^{-\pi})$

$\psi(e^{-\pi})$

$\phi(e^{-\pi})$

$\phi(e^{-3\pi})$

$$= \frac{\pi^{1/4}}{(3/4)}, \quad (6.5)$$

$$= 2^{-5/8} e^{\pi/8} \frac{\pi^{1/4}}{(3/4)}, \quad (6.6)$$

$$= q_4 \; 6\sqrt{3} - 9 . \quad (6.7)$$

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From (3.8) and (6.6), we have

$$M(e^{-\pi/2}) = 2^{-5/4} \frac{\sqrt{\pi}}{{}^2(3/4)}, \quad (6.8)$$

Using (6.8) and (6.2), we obtain

$$M(e^{-\pi}) = \frac{\sqrt{\pi}}{{}^2(3/2)} . \quad (6.9)$$

Setting (6.9) in (6.4), we obtain

$$M(e^{-2\pi}) = \frac{\sqrt{2} - 1}{2} \frac{\sqrt{\pi}}{{}^2(3/2)} . \quad (6.10)$$

J.M. Borwein and P.B. Borwein [7] are the first to observe that class invariant could be used to evaluated certain values of $\phi(e^{-n\pi})$. The Ramanujan Weber class invariants are defined by

$$G_n := 2^{-1/4} q_n^{-1/24} (-q_n; q_n^2)_\infty ,$$

and

$$g_n := 2^{-1/4} q_n^{-1/24} (q_n; q_n^2)_\infty , \quad (6.11)$$

where $q_n = e^{-\pi\sqrt{n}}$. Chan and Huang has derived few explicit formulas for evalu-ating $K(e^{-\pi\sqrt{n/2}})$ in the terms of Ramanujan Weber class. Similar works are done by Adiga ed. al. Analogoues to these works we obtain explicit formulas to evaluate $\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n/2}})}$.

Theorem 6.1. For Ramanujan Weber class invariant defined as in (6.11), let $p = G^{12}_n$ and $p_1 = g_n^{12}$, then

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n/2}})} = \frac{1}{2} \frac{1}{1 q \; p \left(\sqrt{p(p+1)} + \sqrt{p(p-1)}\right)} \, p ,$$

M(e−

M(e−π√ n/2) = 2 rq p1 2 + 1 − p1 .

Proof: From [9], we have

$$G_n = [4\alpha_n(1 - \alpha_n)]^{-1/24} .$$

Hence,

$$\alpha_n = \frac{1}{(\sqrt{p(p+1)} + \sqrt{p(p-1)})^2} .$$

(6.12)(6.13)(6.14)10 Using (6.14) in (6.1), we obtain (6.12).

Also from [9], we have

$$2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n} .$$

Hence

$$\sqrt{\alpha_n} = \sqrt{(p_1^2 + 1)} - p_1 . \quad (6.15)$$

Using (6.15) in (6.1), we complete the proof of (6.13).

Example: Let n = 1. Since $G_1 = 1$, from theorem 6.1 we have

$$\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \frac{1}{2}^{5/4} .$$

Let n = 2. Since $g_2 = 1$, from theorem 6.1 we have

$$\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2}\sqrt{\sqrt{2} - 1} .$$

Remark: Using [10, p. 229] it is easily verified that M(q) and K(q) are related by the equation

$$M(q^2)K(q) + K(q)M(q) - M(q^2) = 0.$$

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